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The Hele-Shaw problem with surface tension in a half-plane: A model problem

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Abstract

Consider the Hele-Shaw problem with surface tension in the half-plane $\{y_1 > 0\}$ when at time $t = 0$ the domain $\Omega(t)$ lies partly on the line $y_1 = 0$, and partly in $\{y_1 > 0\}$. In order to establish existence of a solution to this free boundary problem we need to study the (linear) model problem when the $\Omega(t)$ is a fixed angular domain. In this paper we consider this model problem and establish existence of a solution satisfying sharp weighted Hölder estimates. These estimates will be used in subsequent work to solve the full Hele-Shaw problem.

Keywords: Laplace equation; Free boundary problems; Hele-Shaw problem; Surface tension; Weighted Hölder spaces

1. Introduction

The classical Hele-Shaw problem seeks to determine a fluid domain $\Omega(t)$ and the fluid pressure $p(y, t)$ $(y \in \Omega(t))$ such that

$$\Delta_y p = 0 \quad \text{in } \Omega(t), \tag{1.1}$$

$$p = \kappa \quad \text{on } \partial\Omega(t), \tag{1.2}$$

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$$V_n = -\mu \frac{\partial p}{\partial n} \quad \text{on } \partial\Omega(t), \tag{1.3}$$

where

$$\Omega(0) \quad \text{is given} \tag{1.4}$$

Here κ is the mean curvature, n is the outward normal, V_n is the velocity of the free boundary $\partial\Omega(t)$ in the direction n, and μ is a positive parameter. We will use the sign convention that convex hypersurfaces have positive mean curvature. In particular, we have $\kappa = 1$ for unit sphere. The existence and uniqueness of a solution, for a general smooth initial domain $\Omega(0)$, for small time interval was proved by Chen [4], Escher and Simonett [5], Bazaliy [1], Prokert [8] and Bazaliy and Friedman [2]. The methods used by these authors are all different.

In the present work we consider the situation where $\Omega(t)$ is a two-dimensional domain restricted to lie in the half-plane $\mathbb{R}^2_+ = \{(y_1, y_2); y_1 > 0\}$ and

$$\begin{aligned} &\Omega(0) \subset \mathbb{R}^2_+, \quad \partial\Omega(0) = \Gamma_0(0) \cup \Gamma(0), \\ &\Gamma_0(0) \subset \{y_1 = 0\}, \quad \Gamma(0) \subset \{y_1 > 0\}, \\ &\Gamma_0(0) \neq \phi, \quad \Gamma(0) \neq \phi. \end{aligned} \tag{1.5}$$

Setting

$$\Gamma_0(t) = \partial\Omega(t) \cap \{y_1 = 0\}, \;\; \Gamma(t) = \partial\Omega(t) \cap \{y_1 > 0\}, \tag{1.6}$$

we replace (1.2) and (1.3) by

$$p = \kappa \quad \text{on } \Gamma(t), \tag{1.7}$$

$$V_n = -\mu \frac{\partial p}{\partial n} \quad \text{on } \Gamma(t), \tag{1.8}$$

and

$$-\frac{\partial p}{\partial y_1} = h(y_2) \quad \text{on } \Gamma_0(t). \tag{1.9}$$

Here $h(y_2)$ is a given function, the flux of fluid across the boundary $\Gamma_0(t)$. Setting

$$\Gamma_0(t) = \{y_1 = 0, a(t) < y_2 < b(t)\},$$

where $a(0)$, $b(0)$ are the given end-points of $\Gamma_0(0)$, we assume that

$$h(y_2) = 0 \quad \text{in a neighborhood of the points } a(0), b(0). \tag{1.10}$$

The corresponding problem with no surface tension, i.e., with $p = 0$ on $\Gamma(t)$, was studied by King et al. [7]. They constructed explicit solutions in angular domains and used them to study the motion of the corner points $(0, a(t))$, $(0, b(t))$.

In order to establish local existence (in time) of problem (1.1), (1.4)–(1.10), we shall use the method of Bazaliy and Friedman [2] for problem (1.1)–(1.4) that consist of employing a partition of unity of $\Omega(t)$ followed by local diffeomorphism of $\partial\Omega(t)$ which flattens the free boundary $\Gamma(t)$. The problem is then reduced to the study of an equation of the form $Au = F(u)$ where u is an element in a Banach space, A is a linear operator associated with the local linearized problem about the initial data, and $F(u)$ is a nonlinear operator. We then need to prove that the mapping $u \to A^{-1}F(u)$ has a unique fixed point.

The main difficulty is encountered when dealing with a neighborhood of the corner points $(0, a(0))$, $(0, b(0))$ as we try to derive sharp estimates on solution v of $Av = f$, given f. Here, after flattening the free boundary, v and f are vector-functions defined in an angular domain, in fact a sector, of opening ω (the angle that the tangent to $\overline{\Gamma(0)}$ at $(0, a(0))$ $((0, b(0))$ forms with the positive (negative) y_2-axis). We shall refer to this problem as the “model problem”. In the case of zero surface tension (i.e., $p = 0$ on the free boundary) the model problem was studied by Bazaliy and Vasil’eva [3]. However in the presence of surface tension, the singularity which arises at the corner of the angular domain is much more severe.

In the present paper we establish existence, uniqueness and sharp estimates on the solution u of the model problem $Au = f$. In subsequent work we shall use these results to study the complete problem (1.1), (1.4)–(1.10) and to establish the existence and uniqueness of a solution in a small time interval. The estimates that are derived in the present paper are in terms of weighted Hölder norms.

The structure of the paper is as follows: In §2 we define weighted Hölder spaces and state our main result, Theorem 2.1. To prove this theorem we first derive, in §3, an integral representation for the (linearized) free boundary $\sigma(x_1, t)$. Then, in §4, we estimate $|\sigma|$, and in §§5–7 we estimate the Hölder coefficients of $D^4_{x_1}\sigma$. Using these estimates we prove, in §8, that σ and the corresponding solution u of the Poisson equation form the solution of the model problem as asserted in Theorem 2.1. At the end of §8 we prove a uniqueness theorem. In §9 we finish the proof of Theorem 2.1.

2. Statement of the model problem

We introduce in $\mathbb{R}^2$ cartesian coordinates (y_1, y_2) and polar coordinates (r, φ). Let

$$G = \{(y_1, y_2);\ \ y_1 > 0, \ \ -y_1 \tan\omega < y_2 < 0\},$$

$$g = \{(y_1, y_2);\ \ y_1 > 0,\ \ y_2 = -y_1 \tan\omega\},$$

where $0 < \omega < \frac{\pi}{2}$, and

$$G_T = G \times (0,T), \quad g_T = g \times (0,T), \quad T > 0.$$

We denote by $r(y)$ the distance from a point $y \in G$ to the origin $(0,0)$, and by $r(x,y)$ the minimum $\{r(x), r(y)\}$.

We introduce the Banach space $E_s^{k+\alpha,\beta,\alpha}(G_T)$ of functions u with norm

$$\|u\|_{E_s^{k+\alpha,\beta,\alpha}(G_T)} = \sum_{|\ell|=0}^{k} \Big[\sup_{G_T} r^{|\ell|-s}(y)|D_y^\ell u(y,t)| + \langle D_y^\ell u\rangle^{(\alpha)}_{y,s-|l|,G_T} + \langle D_y^\ell u\rangle^{(\beta)}_{t,s-|l|,G_T} + [D_y^\ell u]^{(\alpha,\beta)}_{s-|l|,G_T} \Big],$$

where

$$\langle v\rangle^{(\alpha)}_{y,s,G_T} = \sup_{(y,t),(x,t) \quad \text{in} \ G_T} r^{\alpha-s}(x,y) \frac{|v(y,t)-v(x,t)|}{|y-x|^\alpha},$$

$$\langle v\rangle^{(\beta)}_{t,s,G_T} = \sup_{(y,t),(y,\tau) \quad \text{in} \ G_T} r^{-s}(y) \frac{|v(y,t)-v(y,\tau)|}{|t-\tau|^\beta}$$

and

$$\big[v\big]^{(\alpha,\beta)}_{s,G_T} = \sup_{(y,t),(x,\tau) \quad \text{in} \ G_T} r^{\alpha-s}(x,y) \times \frac{|v(y,t)-v(x,t)-v(y,\tau)+v(x,\tau)|}{|y-x|^\alpha |t-\tau|^\beta}.$$

In a similar way we introduce the space $E_s^{k+\alpha,\beta,\alpha}(g_T)$.

We will use the space $C_s^{\alpha,\beta}(g_T)$, $\alpha,\beta \in (0,1)$, with norm

$$\|u\|_{C_s^{\alpha,\beta}(g_T)} = \sup_{g_T} |u(x,t)| + \langle u\rangle^{(\alpha)}_{x,s,g_T} + \langle u\rangle^{(\beta)}_{t,s,g_T}.$$

We shall write $u \in M_s^{4+\alpha}(g_T)$ if $u \in E_{s+3}^{4+\alpha,\alpha/3,\alpha}(g_T)$, $u_t \in E_s^{1+\alpha,\alpha/3,\alpha}(g_T)$ and

$$\|u\|_{M_s^{4+\alpha}(g_T)} = \|u\|_{E_{s+3}^{4+\alpha,\alpha/3,\alpha}(g_T)} + \|u_t\|_{E_s^{1+\alpha,\alpha/3,\alpha}(g_T)}.$$

We shall also write $u \in N_s^{4+\alpha}(g_T)$ if $u \in M_s^{4+\alpha}(g_T)$ and

$$D_x^3 u \in C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(g_T) \cap E_s^{1+\alpha,\alpha/3,\alpha}(g_T), \quad D_x^2 u \in C_{s+1}^{\alpha,\frac{2+\alpha}{3}}(g_T) \cap E_{s+1}^{2+\alpha,\alpha/3,\alpha}(g_T).$$

In $N_s^{4+\alpha}(g_T)$ we introduce norm

$$\|u\|_{N_s^{4+\alpha}(g_T)} = \|u\|_{M_s^{4+\alpha}(g_T)} + \left\|D_x^3 u\right\|_{C_{s+2}^{\alpha,\frac{1+\alpha}{3}}(g_T)} + \left\|D_x^2 u\right\|_{C_{s+1}^{\alpha,\frac{2+\alpha}{3}}(g_T)}.$$

The model problem is the following: Find functions $u(y,t)$, $\sigma(r,t)$ such that

$$\begin{aligned}
&\Delta_y u = f_0(y,t) \quad \text{in } G_T,\\
&\frac{\partial \sigma}{\partial t} + \mu \frac{\partial u}{\partial n} = f(y,t) \quad \text{on } g_T,\\
&u + \frac{\partial^2 \sigma}{\partial r^2} = f_1(y,t) \quad \text{on } g_T,\\
&\frac{\partial u}{\partial y_2} = 0 \quad \text{on } y_2 = 0,\\
&\sigma(r,0) = 0.
\end{aligned} \tag{2.1}$$

We assume that

$$\begin{aligned}
&f_0 \in E_{s-1}^{\alpha,\beta,\alpha}(G_T),\ f \in E_s^{1+\alpha,\beta,\alpha}(g_T),\ f_1 \in E_{s+1}^{2+\alpha,\beta,\alpha}(g_T)\\
&\text{for some real number } s > 0,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&f_0 = 0,\ \ f = 0,\ \ f_1 = 0 \text{ if either } t < 0,\ \text{or } t > T_0 \text{ for } 0 < T_0 < T,\\
&\text{or } |y| > R_0 \text{ for some } R_0 > 0.
\end{aligned} \tag{2.3}$$

We also assume that $s > 0$ and that ω is small enough so that

$$\min\left(5 + \frac{\pi}{2\omega},\ 2 + \frac{\pi}{\omega}\right) > 3 + s. \tag{2.4}$$

The main result of this paper is the following:

Theorem 2.1. *Let f_0, f and f_1 be as in* (2.2) *and* (2.3) *with $s > 0$ and assume that ω satisfies* (2.4). *Then there exists a unique solution $u(y,t)$, $\sigma(r,t)$ of* (2.1) *with*

$$u \in E_{s+1}^{2+\alpha,\alpha/3,\alpha}(G_T),\ \ \sigma \in N_s^{4+\alpha}(g_T) \tag{2.5}$$

such that

$$\|u\|_{E^{2+\alpha,\alpha/3,\alpha}_{s+1}(G_T)} + \|\sigma\|_{N^{4+\alpha}_s(g_T)} \leqslant C\left\{\|f_0\|_{E^{\alpha,\alpha/3,\alpha}_{s-1}(G_T)} + \|f\|_{E^{1+\alpha,\alpha/3,\alpha}_s(g_T)} + \|f_1\|_{E^{2+\alpha,\alpha/3,\alpha}_{s+1}(g_T)}\right\}, \tag{2.6}$$

where C *is a constant independent of* f_0, f, f_1.

For simplicity, we shall first prove the following theorem:

Theorem 2.1$_1$. *Let* f_0, f *and* f_1 *be as in* (2.2) *and* (2.3) *with* $s > 0$ *and assume that* ω *satisfies* (2.4). *Then there exists a unique solution* $u(y,t)$, $\sigma(r,t)$ *of* (2.1) *with*

$$u \in E^{2+\alpha,\beta,\alpha}_{s+1}(G_T),\ \ \sigma \in E^{4+\alpha,\beta,\alpha}_{s+3}(g_T), \sigma_t \in E^{1+\alpha,\beta,\alpha}_s(g_T) \tag{2.5$_1$}$$

such that

$$\|u\|_{E^{2+\alpha,\beta,\alpha}_{s+1}(G_T)} + \|\sigma\|_{E^{4+\alpha,\beta,\alpha}_{s+3}(g_T)} + \|\sigma_t\|_{E^{1+\alpha,\beta,\alpha}_s(g_T)} \leqslant C\left\{\|f_0\|_{E^{\alpha,\beta,\alpha}_{s-1}(G_T)} + \|f\|_{E^{1+\alpha,\beta,\alpha}_s(g_T)} + \|f_1\|_{E^{2+\alpha,\beta,\alpha}_{s+1}(g_T)}\right\}, \tag{2.6$_1$}$$

where C *is a constant independent of* f_0, f, f_1.

Note that if $\beta = \alpha/3$ then Theorem 2.1$_1$ is weaker than Theorem 2.1, since it does not give an estimate on $D^2_x\sigma$, $D^3_x\sigma$. In a subsequent paper on the Hele-Shaw problem in a half-plane we shall only need to consider $\beta = \alpha/3$. The proof of Theorem 2.1$_1$ for any β is the same as the proof for $\beta = \alpha/3$, and for notational clarity we have taken here general β.

The proof of Theorem 2.1$_1$ is given in Sections 3–8 and the proof of Theorem 2.1 is given in Section 9.

At the end of Section 8 we shall establish a uniqueness theorem in an appropriate class of solutions which satisfy (2.5), but not necessarily (2.6).

For the sake of clarity, we shall first prove the theorem in the special case where

$$f_0 \equiv 0,\ \ f_1 \equiv 0; \tag{2.5}$$

later on we show how to reduce the general case to this special case.

Introducing the change of variables

$$r = e^{-x_1},\ \ \varphi = x_2 \tag{2.6}$$

and denoting, for simplicity, the functions u, σ, f in the new variables x_1, x_2 again by u, σ, f, system (2.1) becomes:

$$\Delta_x u = 0 \quad \text{in } B_T, \tag{2.7}$$

$$\frac{\partial \sigma}{\partial t} - \mu e^{x_1} \frac{\partial u}{\partial x_2} = f(x_1, t) \quad \text{on } b_T, \tag{2.8}$$

$$u + e^{2x_1}(\sigma_{x_1 x_1} + \sigma_{x_1}) = 0 \quad \text{on } b_T, \tag{2.9}$$

$$\left.\frac{\partial u}{\partial x_2}\right|_{x_2=0} = 0, \ \ \sigma(x_1, 0) = 0, \tag{2.10}$$

where

$$\begin{aligned} &B = \{(x_1, x_2); -\infty < x_1 < \infty, \ \ -\omega < x_2 < 0\}, \ \ B_T = B \times (0, T), \\ &b = \{(x_1, -\omega); \ \ -\infty < x_1 < \infty\}, \ \ b_T = b \times (0, T) \end{aligned} \tag{2.11}$$

and

$$f(x_1, t) = 0 \quad \text{if } x_1 < -\ln R_0 \text{ or } t > T_0. \tag{2.12}$$

We introduce the Banach space $C^{k+\alpha,\beta,\alpha}(B_T)$ of functions $u(x, t)$ with norm

$$\begin{aligned} \|u\|_{C^{k+\alpha,\beta,\alpha}(B_T)} = \sum_{|\ell|=0}^{k} &\Big[\sup_{B_T} |D_x^\ell u| + \langle D_x^\ell u\rangle_{x,B_T}^{(\alpha)} \\ &+ \langle D_x^\ell u\rangle_{t,B_T}^{(\beta)} + \big[D^\ell u\big]_{B_T}^{(\alpha,\beta)}\Big], \end{aligned}$$

where $\langle\cdot\rangle_{x,B_T}^{(\alpha)}$ and $\langle\cdot\rangle_{t,B_T}^{(\beta)}$ are the Hölder constants with respect to x and t, respectively, and

$$[u]_{B_T}^{(\alpha,\beta)} = \sup_{(y,t),(x,\tau) \quad \text{in } B_T} \frac{|u(y, t) - u(x, t) - u(y, \tau) + u(x, \tau)|}{|y - x|^\alpha |t - \tau|^\beta}.$$

In a similar way we introduce the Banach space $C^{k+\alpha,\beta,\alpha}(b_T)$. One can readily check that

$$\begin{aligned} &u(y, t) \in E_s^{k+\alpha,\beta,\alpha}(G_T) \quad \text{if and only if} \\ &e^{sx_1} u(y(x), t) \in C^{k+\alpha,\beta,\alpha}(B_T). \end{aligned} \tag{2.13}$$

Similarly,

$$
\begin{aligned}
&v(y,t) \in E_s^{k+\alpha,\beta,\alpha}(g_T) \text{ if and only if} \\
&e^{sx_1} v(y(x),t) \in C^{k+\alpha,\beta,\alpha}(b_T). \qquad (2.14)
\end{aligned}
$$

In the sequel we shall study system (2.9)–(2.14) and then make use of (2.15) and (2.16).

3. Integral representation for σ

We assume that (u, σ) is a solution of (2.9)–(2.13) and derive an integral representation for σ. Later on we shall prove that this representation yields the solution asserted in Theorem 2.1. We denote by $\tilde{v}(\lambda, x_2, t)$ the Fourier transform of $v(x_1, x_2, t)$ and by $\widehat{w}(\cdot, v)$ the Laplace transform of $w(\cdot, t)$. Then

$$
\frac{\partial^2 \tilde{u}}{\partial x_2^2} - \lambda^2 \tilde{u} = 0, \quad -\omega < x_2 < 0,
$$

$$
\frac{\partial \tilde{u}}{\partial x_2} = 0, \quad x_2 = 0,
$$

$$
\mu \frac{\partial \tilde{u}}{\partial x_2} = \frac{\partial}{\partial t}\, \tilde{\sigma}(\lambda - i, t) - \tilde{f}(\lambda - i, t), \quad x_2 = -\omega \ \text{(by (2.10))}, \qquad (3.1)
$$

$$
\tilde{u}(\lambda - 2i, -\omega, t) = -(-\lambda^2 + i\lambda)\tilde{\sigma}(\lambda, t) \quad \text{(by (2.11))}. \qquad (3.2)
$$

We conclude that

$$
\tilde{u} = M(\lambda, t) \cosh \lambda x_2
$$

where, by (3.1),

$$
M(\lambda, t) = -\frac{1}{\mu\lambda \sinh \lambda\omega} \left(\frac{\partial \tilde{\sigma}}{\partial t}(\lambda - i, t) - \tilde{f}(\lambda - i, t) \right).
$$

Hence

$$
\tilde{u}(\lambda, x_2, t) = -\left[\frac{\partial \tilde{\sigma}(\lambda - i, t)}{\partial t} - \tilde{f}(\lambda - i, t) \right] \frac{\cosh \lambda x_2}{\mu\lambda \sinh \lambda\omega}.
$$

Substituting this into (3.2) we obtain a differential equation for $\tilde{\sigma}$:

$$\frac{\partial\tilde{\sigma}(\lambda-3i,t)}{\partial t} = -\mu(\lambda^2-i\lambda)(\lambda-2i)\big[\tanh(\lambda-2i)\omega\big]\tilde{\sigma}(\lambda,t) + \tilde{f}(\lambda-3i,t). \tag{3.3}$$

Taking the Laplace transform we get

$$v\sigma^*(\lambda-3i,v) = -\mu(\lambda^2-i\lambda)(\lambda-2i)[\tanh(\lambda-2i)\omega]\sigma^*(\lambda,v) + f^*(\lambda-3i,v) \tag{3.4}$$

where we used the notation "$*$" instead of "$\widehat{\sim}$".

We introduce a change of variable:

$$\lambda=-3iz,\quad \sigma^*(-3iz,v)=c(z,v),\quad f^*(-3iz-3i,v)=F(z,v). \tag{3.5}$$

Then (3.4) takes the form

$$vc(z+1,v)+3\mu z(3z+1)(3z+2)[\tan(3z+2)\omega]c(z,v) = F(z,v). \tag{3.6}$$

A similar functional equation with a shift in the argument of the unknown function was considered by Solonnikov and Frolova [9] in connection with the third boundary condition for the Laplace equation in a sector.

We first consider the homogeneous equation, where $F\equiv 0$ in (3.6). Introducing the sequences

$$\alpha_n=\left(\frac{(2n-1)\pi}{\omega}-4\right)\Big/6,\quad \beta_n=\left(\frac{(2n-1)\pi}{\omega}+4\right)\Big/6,$$
$$\rho_n=\left(\frac{n\pi}{\omega}-2\right)\Big/3,\quad \gamma_n=\left(\frac{n\pi}{\omega}+2\right)\Big/3$$

and recalling the formula

$$\frac{\tan z}{z} = \prod_{n=1}^{\infty}\frac{1-z^2/n^2\pi^2}{1-4z^2/(2n-1)^2\pi^2} = \prod_{n=1}^{\infty}\frac{(n\pi-z)(n\pi+z)}{((2n-1)\pi-2z)((2n-1)\pi+z)}\,\frac{(2n-1)^2}{n^2},$$

we can rewrite (3.6) with $F(z) \equiv 0$ and $c = c_0$ in the form

$$\begin{aligned} v c_0(z+1, v) \;=\; & -3^4 \mu\omega \; z\left(z+\tfrac{1}{3}\right)\left(z+\tfrac{2}{3}\right)^2 \frac{\tan 2\omega}{2\omega} \prod_{n=1}^{\infty} \frac{(\rho_n - z)(\gamma_n + z)}{(\alpha_n - z)(\beta_n + z)} \\ & \times \frac{\alpha_n \beta_n}{\rho_n \gamma_n} \; c_0(z, v). \end{aligned} \tag{3.7}$$

When $\omega = \frac{\pi}{4}$ we formally have $\alpha_1 = 0$ and $\tan 2\omega = 0$. We then replace $(\tan 2\omega)\alpha_1$ by $\lim\limits_{\omega \to \frac{\pi}{4}} (\tan 2\omega)\alpha_1 = \frac{8}{\pi}$. Using the property $\Gamma(z+1) = z\Gamma(z)$ of the gamma function we can easily check that the following function is a solution of (3.7):

$$c_0(z, v) = v^{-z+1/2} e^{i\pi z} (3^3 \mu\omega)^{z-1/2} \; \Gamma(z)\Gamma\left(z+\tfrac{1}{3}\right)\Gamma\left(z+\tfrac{2}{3}\right) A(z) E(z), \tag{3.8}$$

where

$$\begin{aligned} A(z) \;=\; & \left(3 \frac{\tan 2\omega}{2\omega}\right)^{z-1/2} \Gamma\left(z+\tfrac{2}{3}\right) \prod_{n=1}^{\infty} \frac{\Gamma(\gamma_n + z)\Gamma(1+\alpha_n - z)}{\Gamma(1+\rho_n - z)\Gamma(\beta_n + z)} \\ & \times \left(\frac{\alpha_n \beta_n}{\gamma_n \rho_n}\right)^{z-1/2} W(n), \end{aligned} \tag{3.9}$$

$$\begin{aligned} W(n) \;=\; & \exp\{\beta_n(\ln \beta_n - 1) + \rho_n(\ln \rho_n - 1) + \gamma_n(1 - \ln \gamma_n) \\ & + \alpha_n(1 - \ln \alpha_n)\}, \end{aligned} \tag{3.10}$$

and $E(z)$ is an arbitrary analytic function such that $E(z+1) = E(z)$. If $\omega \geqslant \frac{\pi}{4}$ then $\alpha_r \leqslant 0$ and $W(1)$ is not defined. In this case we simply define $W(1) = 1$ in (3.9). The convergence of the infinite product in (3.9) can be proved by calculation similar to those in [9].

For our purposes we choose

$$E(z) = \sin^3 \pi z e^{-3i\pi z} \sin^3 \pi\left(z+\tfrac{1}{3}\right) e^{3i\pi(z+1/3)} \sin^2 \pi\left(z+\tfrac{2}{3}\right). \tag{3.11}$$

The function $E(z)$ is chosen so as to eliminate the poles of the functions $\Gamma(z)$, $\Gamma(z+1/3)$, $\Gamma(z+2/3)$ and at the same time not to introduce simple roots. The multiple roots of this function are located at the points $z = \pm n$, $z = \pm n - \frac{1}{3}$, $z = \pm n - \frac{2}{3}$ ($n = 0, 1, 2, \ldots$).

Lemma 3.1. *There holds*

$$A(z) = (\tan(3z+2)\omega)^{z-1/2} e^{\frac{4}{3}\ln z + O(1)} \tag{3.12}$$

as $|\mathrm{Im}\, z| \to \infty$ *while* $|\mathrm{Re}\, z|$ *remains bounded*.

The proof, which is quite lengthy, is based on the asymptotic formula, for $|\mathrm{Im}\, z| \to \infty$ while $|\mathrm{Re}\, z|$ remains bounded,

$$\Gamma(z) = \exp\left\{\left(z - \tfrac{1}{2}\right)\ln z - z + \tfrac{1}{2}\ln 2\pi + \tfrac{1}{12z} + O\left(\tfrac{1}{z^3}\right)\right\}. \tag{3.13}$$

The heuristic arguments to prove Lemma 3.1 is as follows. The function $A\,(z)$ is a solution of the equation

$$c\,(z+1, v) + [\tan\,(3z+2)\,\omega]\, c\,(z, v) = 0$$

and hence the function $\tan\,(3z+2)\,\omega$ here is similar to the function $\omega\,(q)$ from [9]. From Proposition 5.2 [9] and from (5.9) of [9] it follows that a solution of (5.1) in [9] has the asymptotic representation

$$D_0\,(q) \sim \omega\,(q)^{q-1/2}\, M\,(q)\,.$$

So we can expect that in our case

$$A\,(z) \sim (\tan\,(3z+2)\,\omega)^{z-1/2}\, N\,(z)\,.$$

To prove this correctly we use approach from Proposition (5.2) of [9] and get the representation

$$\begin{aligned} A\,(z) \;=\; & (2\pi)^{1/2} \exp\left[(z-1/2)\, Q\,(z)\right] \\ & \times \exp\left[\tfrac{2}{3}\ln\left(z+\tfrac{2}{3}\right) - \left(z+\tfrac{2}{3}\right) + \frac{1}{12\left(z+\frac{2}{3}\right)} + \varsigma_1\left(z+\tfrac{2}{3}\right)\right] \\ & \times \exp\left[Q_1\,(z) + Q_2\,(z) + Q_3\,(z)\right], \end{aligned}$$

where $\varsigma_1(x) = O(x^{-3})$, $Q_k\,(z)$, $(k = 1, 2, 3)$, are estimated similarly to the corresponding functions from [9] such that $Q_1\,(z) \approx \frac{2}{3}\ln z + O(1)$, $Q_2(z) \approx O(1)$, $Q_3(z) \approx O(1)$, and $Q(z) = \tan\,(3z+2)\,\omega$.

We now return to the inhomogeneous equation (3.6) and seek a solution of the form

$$c(z,v) = c_0(z,v)y(z,v).$$

Dropping for brevity the variable v, we see that $y(z)$ needs to satisfy the equation

$$y(z+1) - y(z) = \frac{F(z,v)}{vc_0(z+1,v)} \equiv G(z,v). \tag{3.14}$$

In order to solve (3.14) we first observe that the zeros of $A(z+1)$ are simple zeros located at

$$z_1 = m + \rho_n = m + \frac{n\pi}{3\omega} - \frac{2}{3}$$

and

$$z_2 = -m - 1 - \beta_n = -m - 1 - \frac{(2n-1)\pi}{6\omega} - \frac{2}{3},$$

where $m = 0, 1, 2, \ldots, n = 1, 2, \ldots$. Hence, since $\omega < \pi/2$,

$$\max z_2 = -1 - \frac{\pi}{6\omega} - \frac{2}{3} < -2, \quad \min z_1 = \frac{\pi}{3\omega} - \frac{2}{3} > 0.$$

The zeros of $E(z+1)$ are all multiple zeros, located at

$$\pm n, \quad \pm n - \tfrac{1}{3}, \quad \pm n - \tfrac{2}{3} \ (n = 0, 1, 2, \ldots). \tag{3.15}$$

From (2.14) and the uniform boundedness of $f(x_1,t)$ and $e^{sx_1}f(x_1,t)$ (see (2.16)) it follows that

$$f^*(\lambda, v) \text{ is analytic in } \lambda \text{ for } \operatorname{Im}\lambda < s. \tag{3.16}$$

Since $s > 0$, $F(z,v)$ is analytic in z for $\operatorname{Re} z > -1 - \frac{s}{3}$. Hence the function $G(z,v)$ is analytic and bounded in a strip

$$S_{\varepsilon_0} : -1 - \varepsilon_0 < \operatorname{Re} z < \varepsilon_0$$

for some small ε_0, except for a finite number of poles from the sequences in (3.15), namely, $0, -\frac{1}{3}, -\frac{2}{3}, -1$.

We introduce in this strip the function

$$K(\zeta) = \frac{1}{2i}(\cot \pi\zeta + i)\ \frac{\sin^8 \pi a}{\sin^8 \pi(\zeta + a)} \quad (0 < a < 1). \tag{3.17}$$

$K(\zeta)$ is periodic of period 1, with simple poles at $\zeta = 0, \pm 1$ and multiple poles at $\zeta = -a$.

We seek a solution to (3.14) in the form

$$y(z) = \int_{L_{\delta_0}} G(z + \zeta, v) K(\zeta)\, d\zeta, \tag{3.18}$$

where the contour L_{δ_0} consists of three parts: the interval $\{\mathrm{Re}\,\zeta = -1,\ -\infty < \mathrm{Im}\,\zeta < -\delta_0\}$, the half-circle $\{|\zeta + 1| = \delta_0, \mathrm{Re}\,\zeta > -1\}$, and the interval $\{\mathrm{Re}\,\zeta = -1,\ \delta_0 < \mathrm{Im}\,\varsigma < \infty\}$, where δ_0 is any positive number smaller than $\varepsilon_0/2$.

Remark 3.1. The factor $\sin^8 \pi(\zeta + a)$ in (3.17) is needed not only to ensure the convergence of the integral in (3.18), but also to make the function H in (3.25) decrease fast enough to zero as $|y| \to \infty$. This latter property will be used a number of times; for instance, in asserting that the function $\Phi(y, y - k_1, \xi)$ defined in (4.11) and its derivative $\partial\Phi/\partial y$ vanish at $y = +\infty$.

Lemma 3.2. *The function* $y(z)$ *defined in* (3.18) *is a solution of* (3.14) *in the strip* $-\frac{1}{2}\varepsilon_0 < \mathrm{Re}\, z < 1 + \frac{1}{2}\varepsilon_0$.

Proof. Using the asymptotic formulas (3.12), (3.13) and

$$v^{-z+1/2} \approx e^{(\arg v)\mathrm{Im}\, z}$$

as $\mathrm{Im}\, z \to \pm\infty$, we deduce from (3.8) and (3.11) that

$$\left|\frac{1}{c_0(z, v)}\right| \leqslant const.\, e^{-6.5\pi|\mathrm{Im}\, z|}|z|^{c_1} \text{ as } |\mathrm{Im}\, z| \to \infty,$$

for some constant $c_1 \geqslant 0$. Hence the function $G(z, v)$ satisfies

$$|G(z, v)| \leqslant const.\, e^{-6.5\pi|\mathrm{Im}\, z|}|z|^{c_1} \text{ as } |\mathrm{Im}\, z| \to \infty. \tag{3.19}$$

It follows that the integral in (3.18) is well defined, not only for the above contour L_{δ_0} but also for any contour $\mathrm{Re}\,\zeta = -\delta$ $(0 < \delta < 1)$ and $-\varepsilon_0 < z + \delta < 1$ provided $-\delta$ avoids the points $-\frac{1}{3}, -\frac{2}{3}$. All such contour integrals yield the same function $y(z)$; this fact will be used later on.

We can now establish relation (3.14) rather easily by changing the contour of integration in the integral representation (3.18) for $y(z+1)$, using (3.17); this is the same argument as in Proposition 6.1 of [9].

We have shown so far that the function

$$c(z, v) = c_0(z, v) \int_{L_{\delta_0}} G(z+\zeta, v) K(\zeta)\, d\zeta \tag{3.20}$$

is a solution of (3.6). The next step is to take the inverse Laplace and Fourier transforms of this formula in order to derive an explicit integral representation for $\sigma(x, t)$ on b_T. But we first want to rewrite $c(z, v)$ in a more convenient way. We shall use the fact, noted above, that we may replace L_{δ_0} by the line $\zeta = -\delta + iy$, for any small $\delta > 0$. Since

$$\begin{aligned}
&\frac{c_0(z, v)}{v c_0(z+\zeta+1, v)} K(\zeta)\\
&\quad = \frac{1}{2i}(\operatorname{ctg} \pi\zeta + i) \frac{\sin^8 \pi a}{\sin^8 \pi(\zeta + a)}\\
&\qquad \times \frac{v^{-z+1/2}(3^3\mu\omega)^{z-1/2} e^{i\pi z}}{v v^{-(z+\zeta+1)+1/2}(3^3\mu\omega)^{(z+\zeta+1)-1/2} e^{i\pi(z+\varsigma+1)}}\\
&\qquad \times \frac{\Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3)A(z)}{\Gamma(z+\zeta+1)\Gamma(z+\zeta+1+1/3)\Gamma(z+\zeta+1+2/3)A(z+\zeta+1)}\\
&\qquad \times \frac{\sin^3 \pi z e^{-3i\pi z} \sin^3 \pi(z+1/3) e^{3i\pi(z+1/3)}}{\sin^3 \pi(z+\zeta+1) e^{-3i\pi(z+\zeta+1)} \sin^3 \pi(z+\zeta+1+1/3) e^{3i\pi(z+\zeta+1+1/3)}}\\
&\qquad \times \frac{\sin^2 \pi(z+2/3)}{\sin^2 \pi(z+\zeta+1+2/3)}\\
&\quad = \frac{-1}{e^{i\pi\zeta} - e^{-i\pi\zeta}} \; \frac{\sin^8 \pi a}{\sin^8 \pi(\zeta+a)} v^{\zeta} (3^3\mu\omega)^{-\zeta-1} \frac{L(z)}{L(z+\zeta+1)},
\end{aligned}$$

where

$$L(z) = \sin^3 \pi z \sin^3 \pi \left(z + \tfrac{1}{3}\right) \sin^2 \pi \left(z + \tfrac{2}{3}\right) \Gamma(z)\Gamma\left(z+\tfrac{1}{3}\right)\Gamma\left(z+\tfrac{2}{3}\right) A(z), \tag{3.21}$$

we get

$$\begin{aligned}
c(z, v) = {}& \frac{1}{2}\int_{-\infty}^{\infty} dy\, v^{-\delta+iy} (3^3\mu\omega)^{-1+\delta-iy} \frac{\sin^8 \pi a}{\sin \pi(\delta - iy) \sin^8 \pi(-\delta + iy + a)}\\
&\times \frac{L(z)}{L(z+1-\delta+iy)} F(z-\delta+iy, v).
\end{aligned} \tag{3.22}$$

Hence

$$\sigma^*(\lambda, v) = \frac{1}{2}\int_{-\infty}^{\infty} dy \; \frac{v^{-\delta+iy}(3^3\mu\omega)^{(-1+\delta-iy)}\sin^8 \pi a}{\sin \pi(\delta - iy)\sin^8 \pi(-\delta + iy + a)}$$
$$\times \frac{L\left(-\frac{i\lambda}{\gamma}\right)}{L\left(-\frac{i\lambda}{\gamma} + 1 - \delta + iy\right)} \; f^*(\lambda - \gamma(y + i\delta - i), v), \tag{3.23}$$

where $\gamma = -3$.

From the equality

$$\int_0^{\infty} t^{\ell-1}e^{-vt}\, dt = \frac{1}{v^{\ell}}\Gamma(\ell)$$

we get, by the formula for the inverse Laplace transform:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} v^{-\delta+iy}e^{vt}\, dv = \frac{t^{-1+\delta-iy}}{\Gamma(\delta - iy)}.$$

Hence, taking the inverse Laplace transform in (3.23) we get

$$\tilde{\sigma}(\lambda, t) = \frac{1}{2}\int_0^t \frac{d\tau}{(-\gamma^3(t-\tau))^{1-\delta}}\int_{-\infty}^{\infty} dy\, e^{-iy\ln(-\gamma^3(t-\tau))}H(y, \delta)$$
$$\times \frac{L\left(-\frac{i\lambda}{\gamma}\right)}{L\left(-\frac{i\lambda}{\gamma} + 1 - \delta + iy\right)}\tilde{f}(\lambda - \gamma(y + i\delta - i), \tau), \tag{3.24}$$

where

$$H(y, \delta) = \frac{(-\mu\omega)^{(-1+\delta-iy)}\sin^8 \pi a}{\Gamma(\delta - iy)\sin \pi(\delta - iy)\sin^8 \pi(-\delta + iy + a)}. \tag{3.25}$$

Since the inverse Fourier transform of $\tilde{f}(\lambda - \gamma(y + i\delta - i), \tau)$ is $f(x, \tau)e^{\gamma(1-\delta)x}e^{i\gamma yx}$, taking the inverse Fourier transform of (3.24), we end up with the formula

$$\sigma(x, t) = \frac{1}{4\pi}\int_0^t \frac{d\tau}{(-\gamma^3(t-\tau))^{1-\delta}}\int_{-\infty}^{\infty} d\xi f(x-\xi, \tau)e^{\gamma(1-\delta)(x-\xi)}$$
$$\times \int_{-\infty}^{\infty} dy \; e^{-iyq(t-\tau,x)}H(y, \delta)\int_{-\infty}^{\infty} \frac{L\left(-\frac{i\lambda}{\gamma}\right)}{L\left(-\frac{i\lambda}{\gamma} + 1 - \delta + iy\right)}$$
$$\times e^{i\lambda\xi - i\gamma y\xi}\, d\lambda, \tag{3.26}$$

where

$$q(t, x) = \ln(-\gamma^3 t) - \gamma x; \tag{3.27}$$

here, for brevity, x denotes the variable x_1.

We finally want to transform the inner integral in (3.26) into a more useful form. To do that we first examine the zeros and poles of the function $L(z)$. From (3.21) and (3.9) we see that the simple zeros of $L(z)$ are given by

$$1 + \rho_n - z = -m \text{ and } \beta_n + z = -m \quad (m = 0, 1, 2, \ldots).$$

Hence $L(z)$ has no simple zeros in the strip

$$-\frac{\pi}{6\omega} - \frac{2}{3} < \operatorname{Re} z < 1 + \frac{\pi}{3\omega} - \frac{2}{3}.$$

The multiple zeros of $L(z)$ are

$$z = \pm m, \quad \pm m - \tfrac{1}{3}, \; n - \tfrac{2}{3} \quad (m = 0, 1, 2 \ldots, n = 1, 2, \ldots). \tag{3.28}$$

The poles of $L(z)$ are given by

$$\gamma_n + z = -m, \; 1 + \alpha_n - z = -m \quad (m = 0, 1, 2, \ldots)$$

and they all lie outside the strip

$$-\frac{2}{3} - \frac{\pi}{3\omega} < \operatorname{Re} z < \frac{1}{3} + \frac{\pi}{6\omega}.$$

Setting

$$v_1 = \max\left\{\frac{2}{3} - \delta - \frac{\pi}{3\omega}, -\frac{1}{3} - \frac{\pi}{6\omega}\right\} < 0,$$
$$v_2 = \min\left\{\frac{5}{3} - \delta + \frac{\pi}{6\omega}, \frac{2}{3} + \frac{\pi}{3\omega}\right\} > 0 \tag{3.29}$$

we conclude that

$$\frac{L(i(k-y))}{L(ik+1-\delta)} \quad \text{has only poles with multiplicity } > 1$$
$$\text{in the strip } v_1 < \operatorname{Im} k < v_2. \tag{3.30}$$

We next evaluate $L(im+d)$ as $|m| \to \infty$ for real m and d. By (3.13)

$$\Gamma\,(im+d) \approx const.\, e^{-|m|\frac{\pi}{2}}\,|im+d|^{d-1/2}\, e^{im\ln|im+d|-(im+d)},$$

in addition $\sin\pi\,(im+d) \approx const.\, e^{|m|\pi}$, so that

$$\Gamma(im+d)\sin\pi(im+d) \approx const.\, e^{\frac{\pi}{2}|m|+im\ln(im+d)-(im+d)}|im+d|^{d-\frac{1}{2}}$$

as $|m| \to \infty$. From the equality

$$\tan\left(\alpha+i\beta\right) = \frac{\sin 2\alpha + i\sinh 2\beta}{\cos 2\alpha + i\cosh 2\beta},$$

we get

$$\left|\tan\left(\alpha+i\beta\right)\right| \leqslant const. \text{ for } \left|\beta\right| \to \infty,$$

$$\arg\tan\left(\alpha+i\beta\right) \to \pm\frac{\pi}{2}\ as\ \beta \to \infty.$$

From this it follows for $|m| \to \infty$

$$\left[\tan\left(\alpha+i\beta\right)\right]^{im+d-1/2} \approx const.\, e^{-|m|\frac{\pi}{2}}$$

and by (3.12) we get

$$A(im+d) \approx const.\, e^{\left\{-\frac{\pi}{2}|m|+O(1)\right\}}\cdot|im+d|^{4/3}. \tag{3.31}$$

Hence

$$\begin{aligned} L(im+d) \ \approx\ & const.\, e^{6\pi|m|+im\left(\ln|im+d|\ \ |im+d+\frac{1}{3}|\ \ |im+d+\frac{2}{3}|\right)} \\ & \times|im+d|^{d-\frac{1}{2}}|im+d+\tfrac{1}{3}|^{d-\frac{1}{2}+\frac{1}{3}} \\ & \times|im+d+\tfrac{2}{3}|^{d-\frac{1}{2}+\frac{2}{3}}|im+d|^{\frac{4}{3}} \end{aligned} \tag{3.32}$$

as $|m| \to \infty$. Setting

$$k = k_1+ik_2,\ \ m_1 = k_1-y,\ \ m_2 = k_1,\ \ d_1 = -k_2,\ \ d_2 = -k_2+1-\delta,$$

we get

$$E(k,y,\delta) \equiv \frac{L(i(k-y))}{L(ik+1-\delta)} \approx const.\, \frac{e^{6\pi|k_1-y|}}{e^{6\pi|k_1|}} N(k,y,\delta)R(k,y,\delta),$$

where

$$N(k,y,\delta) = \frac{e^{i(k_1-y)\ln|im_1+d_1|\ \ |im_1+d_1+\frac{1}{3}|\ \ |im_1+d_1+\frac{2}{3}|} e^{-4(im_1+d_1)}}{e^{ik_1\ln|im_2+d_2|\ \ |im_2+d_2+\frac{1}{3}|\ \ |im_2+d_2+\frac{2}{3}|} e^{-4(im_2+d_2)}},$$

$$R(k,y,\delta) = \frac{|im_1+d_1|^{d_1-\frac{1}{2}}|im_1+d_1+\frac{1}{3}|^{d_1-\frac{1}{2}+\frac{1}{3}}|im_1+d_1+\frac{2}{3}|^{d_1-\frac{1}{2}+\frac{2}{3}}}{|im_2+d_2|^{d_2-\frac{1}{2}}|im_2+d_2+\frac{1}{3}|^{d_2-\frac{1}{2}+\frac{1}{3}}|im_2+d_2+\frac{2}{3}|^{d_2-\frac{1}{2}+\frac{2}{3}}} \times \frac{|im_1+d_1|^{\frac{4}{3}}}{|im_2+d_2|^{\frac{4}{3}}}. \tag{3.33}$$

We easily see that

$$|N(k,y,\delta)| \leqslant const., \quad |R(k,y,\delta)| \leqslant \frac{const.}{|k_1|^{3(1-\delta)}} \tag{3.34}$$

as $|k_1| \to \infty$ and for bounded y, for example, so that in this case

$$\left| \frac{L(i(k-y))}{L(ik+1-\delta)} \right| \leqslant \frac{const.}{|k_1|^{3(1-\delta)}}. \tag{3.35}$$

Recalling also (3.30) we deduce that the contour integral

$$\int_{-\infty}^{\infty} \frac{L(-i(k-y))}{L(ik+1-\delta)} e^{-i\gamma\xi k_1}\, dk_1 \text{ is independent of } k_2,\ \nu_1 < k_2 < \nu_2. \tag{3.36}$$

We now substitute $\lambda = -\gamma(k-y)$ $\left(\frac{1}{3}\nu_1 < \operatorname{Im}\lambda < \frac{1}{3}\nu_2\right)$ into the inner integral in (3.26) and use (3.36) to conclude that

$$\int_{-\infty}^{\infty} \frac{L\left(-\frac{i\lambda}{\gamma}\right)}{L\left(-\frac{i\lambda}{\gamma}+1-\delta+iy\right)}\ e^{i\lambda\xi - i\gamma y\xi}\, d\lambda$$

$$= -\gamma e^{\gamma\xi k_2} \int_{-\infty}^{\infty} \frac{L(i(k_1-y)-k_2)}{L(ik_1-k_2+1-\delta)}\ e^{-i\gamma\xi k_1}\, dk_1$$

for any $k_2 \in (v_1, v_2)$. Hence (3.26) takes the form

$$\sigma(x,t) = const. \int_0^t \frac{d\tau}{(t-\tau)^{1-\delta}} \int_{-\infty}^{\infty} d\xi \varphi(x-\xi,\tau) e^{(-3(1-\delta)-s)x} e^{(3(1-\delta)+s)\xi} e^{-3k_2\xi}$$
$$\times \int_{-\infty}^{\infty} dy\, e^{-iyq(t-\tau,x)} H(y,\delta) \int_{-\infty}^{\infty} \frac{L(i(k_1-y)-k_2)}{L(ik_1-k_2+1-\delta)} e^{3i\xi k_1}\, dk_1, \quad (3.37)$$

where

$$\varphi(x,t) = f(x,t)e^{sx} \quad (x = x_1). \quad (3.38)$$

It is important to note that k_2 may be chosen to depend on ξ.

4. Estimate on $\sigma(x,t)$

Set

$$B_j(x,\xi,t,\delta) = \int_{a_j}^{b_j} dy\;\, e^{-iyq(t,x)} H(y,\delta) \int_{-\infty}^{\infty} E(k_1+ik_2,y,\delta) e^{3i\xi k_1}\, dk_1, \quad (4.1)$$

where $(a_1,b_1) = (-\infty,0),\;\; (a_2,b_2) = (0,\infty)$, so that

$$\sigma(x,t) = const. \sum_{j=1,i=1}^{2} \int_0^t \frac{d\tau}{(t-\tau)^{1-\delta}} \int_{a_i}^{b_i} d\xi \varphi(x-\xi,\tau) e^{(-3(1-\delta)-s)x} e^{(3(1-\delta)+s)\xi}$$
$$\times e^{-3k_2\xi} B_j(x,\xi,t-\tau,\delta) \equiv \sum_{j=1}^{2} \left(\sigma_{1j}+\sigma_{2j}\right), \quad (4.2)$$

where σ_{1j}, σ_{2j} correspond to B_j. To estimate σ_{2j} we shall choose k_2 near v_2 (but $k_2 < v_2$) and to estimate σ_{1j} we shall choose k_2 negative (but $k_2 > v_1$).

We begin with σ_{22} and first proceed to estimate B_2. Following Bazaliy and Vasil'eva [3] we decompose the domain of integration

$$\Omega = \{(y,k_1):\;\; y > 0, -\infty < k_1 < \infty\}$$

into eight domains, $\Omega = \cup_{j=1}^{8} \Omega_j$, as shown in Fig. 1. Here M is a sufficiently large number such that the terms $O(1/z^3)$ in (3.13) and $O(1)$ in (3.12) can be neglected in all regions Ω_j, except Ω_8.

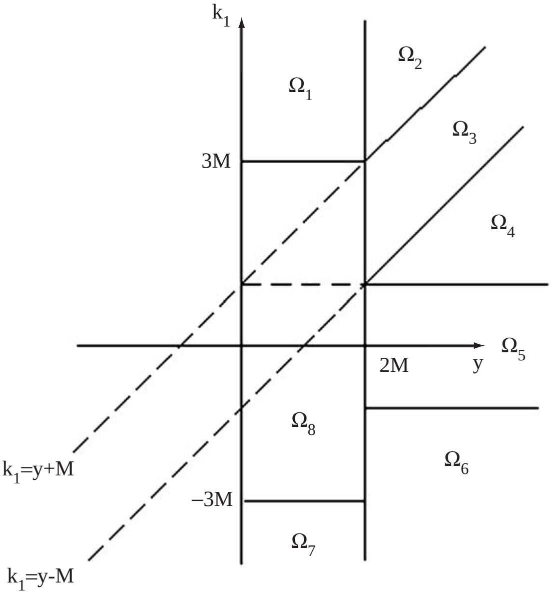

Fig. 1.

Write

$$B_2 = \sum_{j=1}^{8} B_{2j},$$

where in B_{2j} the integration is taken over Ω_j. Consider

$$B_{21} = \int_0^{2M} dy \int_{3M}^{\infty} dk_1 \ldots \ .$$

Since $k_1 > 3M$, $0 < y < 2M$ we have $k_1 - y \geqslant M$ and $|k_1 - y| - |k_1| = -y$. Hence from (3.33) and (3.34) we obtain

$$|E(k_1 + ik_2, y, \delta)| \leqslant const.\, e^{-6\pi y} \frac{1}{k_1^{3(1-\delta)}},$$

and consequently

$$|B_{21}(x, \xi, t, \delta)| \leqslant const.$$

Similarly we can estimate all B_{2j}. Indeed, B_{28} can be estimated using the fact that both k_1 and y are uniformly bounded. To estimate $B_{22}, \ldots, B_{27}$ we use the asymptotic behaviour of the function $H\left(y, \delta\right)$ for large y and estimates of the values E_{2j} corresponding to B_{2j}. We have

$$
\begin{aligned}
|E_{22}| &\leqslant ce^{-6\pi y}\frac{1}{k_1^{3(1-\delta)}}, \ \ (y, k_1) \in \boldsymbol{\Omega}_2,\\
|E_{23}| &\leqslant ce^{-6\pi k_1}\frac{1}{k_1^{-3k_2+5/6+3(1-\delta)}}, \ \ (y, k_1) \in \boldsymbol{\Omega}_3,\\
|E_{24}| &\leqslant ce^{6\pi(y-2k_1)}\left(1+\frac{y^{-3k_2+5/6}}{k_1^{-3k_2+5/6}}\right)\frac{1}{k_1^{3(1-\delta)}}, \ \ (y, k_1) \in \boldsymbol{\Omega}_4,\\
|E_{25}| &\leqslant ce^{6\pi y}y^{-3k_2+5/6}, \ \ (y, k_1) \in \boldsymbol{\Omega}_5,\\
|E_{26}| &\leqslant ce^{6\pi y}\left(1+\frac{y}{|k_1|}\right)^{-3k_2+5/6}\frac{1}{k_1^{3(1-\delta)}}, \ \ (y, k_1) \in \boldsymbol{\Omega}_6,\\
|E_{27}| &\leqslant c\frac{1}{k_1^{3(1-\delta)}}, \ \ (y, k_1) \in \boldsymbol{\Omega}_7,
\end{aligned}
$$

and in addition we notice that $y/\,|k_1| \leqslant y/M$ for $(y, k_1) \in \boldsymbol{\Omega}_6$ so that for $-3k_2+5/6 \geqslant 0$

$$\left(1+\frac{y}{|k_1|}\right)^{-3k_2+5/6} \leqslant cy^{-3k_2+5/6}$$

and for $-3k_2+5/6 \leqslant 0$

$$\left(1+\frac{y}{|k_1|}\right)^{-3k_2+5/6} \leqslant const.$$

Thus altogether we obtain

$$|B_{2j}(x, \xi, t, \delta)| \leqslant const.$$

Hence

$$|\sigma_{22}(x,t)| \leqslant const.\, T^{\delta} e^{(-3(1-\delta)-s)x} |\varphi|_{L^\infty} \tag{4.3}$$

provided the ξ-integral

$$\int_0^\infty d\xi \; e^{(3(1-\delta)+s)\xi} e^{-3k_2\xi}$$

is convergent, i.e., provided

$$3(1-\delta) + s - 3k_2 < 0. \tag{4.4}$$

But in view of assumption (2.4) and the definition of v_2 in (3.29), (4.4) is satisfied if δ and $v_2 - k_2$ are sufficiently small $(k_2 < v_2)$.

To estimate σ_{12} we proceed as before, but now we have to choose k_2 and δ so that

$$3(1-\delta) + s - 3k_2 > 0,$$

and we can do this by taking k_2 to be any negative number $> v_1$ and δ small. Hence estimate (4.3) holds for σ_{12}, and moreover by similar arguments

$$|\sigma(x,t)| \leqslant const.\, T^{\delta} e^{-(s+3-3\delta)x} |\varphi|_{L^\infty}. \tag{4.5}$$

Our next objective is to extend (4.5) to $\delta = 0$. Since the constant in (4.5) depends on δ, we cannot simply take $\delta \to 0$ in (4.5).

Lemma 4.1. *There hold*

$$|\sigma(x,t)| \leqslant const.\, e^{-(s+3)x} |\varphi|_{L^\infty}. \tag{4.6}$$

Proof. Set

$$\begin{aligned} &H(y) = H(y,\delta)|_{\delta=0}, \;\; E(k,y) = E(k,y,\delta)|_{\delta=0}, \\ &B_j(x,\xi,t) = B_j(x,\xi,t,\delta)|_{\delta=0}, \;\; B = B_1 + B_2. \end{aligned} \tag{4.7}$$

Since $\zeta = -\delta + iy = 0$ is a simple pole of the function $K(\zeta)$ in (3.17), by the residue theorem we get

$$\sigma(x,t) = \sigma^0(x,t) + I(x,t), \tag{4.8}$$

where $\sigma^0(x,t)$ is defined as in (4.2) with $\delta = 0$ and with

$$\int_0^t \frac{d\tau}{(t-\tau)^{1-\delta}} \text{ replaced by } \int_{-\infty}^t \frac{d\tau}{t-\tau},$$

i.e. (recalling that $\varphi(x,t)=0$ if $t<0$),

$$\sigma^0(x,t) = const. \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} d\xi \varphi(x-\xi,\tau) e^{(-3-s)(x-\xi)} e^{-3k_2\xi} \times B(x,\xi,t-\tau) \tag{4.9}$$

and

$$I(x,t) = const. \int_{-\infty}^{\infty} \tilde{f}(\lambda + i\gamma, t) \; \frac{L\left(-\frac{i\lambda}{\gamma}\right)}{L\left(-\frac{i\lambda}{\gamma}+1\right)} e^{i\lambda x}\, d\lambda \tag{4.10}$$

is the residue at the pole $\zeta = -\delta + iy = 0$. Since $\int_{-\infty}^{t} \frac{d\tau}{t-\tau} = \infty$, we shall need to estimate the inner integral in (4.9) by a function of $t-\tau$ which will make the τ-integral finite. To do that we write (see (4.1))

$$B(x,\xi,t) = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dy \;\, e^{-iyq(t,x)-b|y|} \Phi(y, y-k_1, \xi),$$

where

$$\Phi(y, y-k_1, \xi) = H(y) \frac{L(i(k_1-y)-k_2)}{L(ik_1-k_2+1)} \; e^{b|y|} e^{3i\xi k_1} \tag{4.11}$$

and b is any sufficiently small positive number.

By integration by parts,

$$\begin{aligned}
&\int_{0}^{\infty} dy \;\, e^{-iyq(t,x)-by} \Phi(y, y-k_1, \xi) \\
&\quad = \int_{0}^{\infty} \frac{1}{-iq-b} \; \frac{\partial}{\partial y}\left(e^{-iyq-by}\right) \Phi \, dy \\
&\quad = \frac{1}{iq+b} \Phi(0,-k_1,\xi) + \frac{1}{iq+b} \; \int_{0}^{\infty} e^{-iyq-by} \frac{\partial \Phi}{\partial y}\, dy \\
&\quad = \frac{1}{iq+b} \Phi(0,-k_1,\xi) + \frac{1}{(iq+b)^2} \; \frac{\partial \Phi}{\partial y} \; (+0,-k_1,\xi) \\
&\qquad + \frac{1}{(iq+b)^2} \; \int_{0}^{\infty} e^{-iyq-by} \frac{\partial^2 \Phi}{\partial y^2}\, dy
\end{aligned}$$

since Φ and Φ_y vanish as $y = +\infty$. Similarly

$$\begin{aligned}
&\int_{-\infty}^{0} dy\ e^{-iyq(t,x)+by}\Phi(y, y-k_1\xi) \\
&= \frac{1}{-iq+b}\Phi(0,-k_1,\xi) \\
&\quad -\frac{1}{(-iq+b)^2}\ \frac{\partial\Phi}{\partial y}(-0,-k_1,\xi) \\
&\quad +\frac{1}{(-iq+b)^2}\int_{-\infty}^{0} e^{-iyq+by}\frac{\partial^2\Phi}{\partial y^2}\,dy.
\end{aligned}$$

Hence

$$\begin{aligned}
B(x,\xi,t) \;=\; & \frac{2b}{b^2+q^2(t,x)}\ \int_{-\infty}^{\infty} dk_1\Phi(0,-k_1,\xi) \\
& +\int_{-\infty}^{\infty}\Bigg[\frac{1}{(iq+b)^2}\ \frac{\partial\Phi}{\partial y}\ (y, y-k_1,\xi)\Bigg|_{y=+0} \\
& -\frac{1}{(-iq+b)^2}\ \frac{\partial\Phi}{\partial y}(y, y-k_1,\xi)\Bigg|_{y=-0}\Bigg]dk_1 \\
& +\frac{1}{(b-iq)^2}\ \int_{-\infty}^{\infty} dk_1\int_{-\infty}^{0}\ e^{-iyq+by}\Phi_{yy}(y, y-k_1,\xi)\,dy \\
& +\frac{1}{(b+iq)^2}\ \int_{-\infty}^{\infty} dk_1\ \int_{0}^{\infty}\ e^{-iyq-by}\Phi_{yy}(y, y-k_1,\xi)\,dy \\
=\; & \sum_{\ell=1}^{4} D_\ell(x,\xi,t),
\end{aligned} \tag{4.12}$$

and we accordingly write

$$\sigma^0(x,t) = \sum_{\ell=1}^{4}\sigma_\ell^0(x,t).$$

Consider σ_4^0 and split the (k_1, y) domain of integration into eight regions Ω_j, as in Fig. 1. Set, accordingly,

$$D_4 = \sum_{j=1}^{8} D_{4j} \text{ and } \sigma_4^0 = \sum_{j=1}^{8}\ \sigma_{4j}^0.$$

Then

$$D_{41} = \frac{1}{(b+iq)^2} \int_0^{2M} dy\ e^{-iyq(t,x)-by} \frac{\partial^2}{\partial y^2}$$
$$\times \left[H(y) \int_{3M}^{\infty} \frac{L(i\,(k_1-y)-k_2)e^{3i\xi k_1}e^{by}}{L(ik_1-k_2+1)}\, dk_1 \right].$$

Proceeding as in the estimates of B_{21} above, but with $\delta = 0$, we find that

$$|D_{41}| \leqslant \frac{const.}{|b+iq(t,x)|^2}.$$

Hence

$$|\sigma_{41}^0| \leqslant const. \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \cdot \frac{1}{|b+iq(t-\tau,x)|^2}$$
$$\times \int_{-\infty}^{\infty} e^{(-3-s)x} e^{(3+s)\xi} e^{-3k_2\xi} |\varphi|_{L^\infty}\, d\xi,$$

where k_2 can be chosen to depend on ξ. Choosing $k_2 < 0$ in the integral $\int_{-\infty}^{0} \dots d\xi$ and k_2 near v_2 in the integral $\int_0^\infty \dots d\xi$, we conclude that

$$|\sigma_{41}^0| \leqslant const.\, e^{(-3-s)x} \int_{-\infty}^{t} \frac{1}{t-\tau} \frac{d\tau}{\left(\ln(-\gamma^3(t-\tau)) - \gamma x\right)^2 + b^2} |\varphi|_{L^\infty}$$
$$\leqslant const.\, |\varphi|_{L^\infty} e^{(-3-s)x}.$$

The other σ_{4j}^0 can be estimated in a similar way, so that

$$|\sigma_4^0| \leqslant const.\, e^{(-3-s)x} |\varphi|_{L^\infty}.$$

Similarly we can estimate σ_1^0, σ_2^0 and σ_3^0 with the result that

$$|\sigma^0(x,t)| \leqslant const.\, e^{(-3-s)x} |\varphi|_{L^\infty}. \tag{4.13}$$

We next estimate $I(x,t)$. From the definition of $L(z)$ we see that

$$I(x,t) = const. \int_{-\infty}^{\infty} \frac{\tilde{f}(\lambda - 3i, t)e^{i\lambda x}}{\frac{i\lambda}{3}\ \left(\frac{i\lambda}{3}+\frac{1}{3}\right)\left(\frac{i\lambda}{3}+\frac{2}{3}\right)\tan(i\lambda+2)\omega}\, d\lambda.$$

Hence

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+1\right)I(x,t) = const.\int_{-\infty}^{\infty} d\xi\varphi(x-\xi,t)e^{-3(x-\xi)-s(x-\xi)} \times\int_{-\infty}^{\infty} \frac{e^{i\lambda\xi}}{(i\lambda+2)\tan(i\lambda+2)\omega}\,d\lambda. \tag{4.14}$$

The denominator in the inner integral has no simple zeros in the strip

$$2-\frac{\pi}{\omega} < \operatorname{Im}\lambda < 2+\frac{\pi}{\omega}. \tag{4.15}$$

The inner integral is convergent if $d\lambda = d\lambda_1$ while $\lambda_2 = \operatorname{Im}\lambda$ is fixed, for any $\lambda_2 \in \left(2-\frac{\pi}{\omega}, 2+\frac{\pi}{\omega}\right)$ since the integrand is then $\approx const.\frac{e^{i\lambda_1\xi}}{\lambda_1}$ for $\left|\lambda_1\right|$ large. Furthermore, the integral is independent of λ_2. Taking λ_2 such that

$$3+s < \lambda_2 < 2+\frac{\pi}{\omega} \text{ if } \xi > 0$$

which is feasible by (2.4), and

$$2-\frac{\pi}{\omega} < \lambda_2 < s+3 \text{ if } \xi < 0, \quad (\omega < \pi/2),$$

which clearly is feasible since $\omega < \pi/2$ and $s > 0$, we deduce that

$$\left|\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+1\right)I(x,t)\right| \leqslant const.\, e^{-(3+s)x}|\varphi|_{L^\infty}. \tag{4.16}$$

From (4.5), (4.13) and (4.8) we see that

$$|I(x,t)| \leqslant const.\, e^{(-s-1+3\delta)x} \quad \text{as } x\to\infty. \tag{4.17}$$

By the mean value theorem $|I_x(x_0,t)| \leqslant const.$ for some point x_0. Writing

$$(I_x+I)(x,t) = (I_x+I)(x_0,t) + \int_{x_0}^{x}(I_{xx}+I_x)\,dx$$

and using (4.16) and (4.17), it follows that $|I_x| \leqslant const.$ and then, by (4.16), also

$$|I_{xx}| \leqslant const. \tag{4.18}$$

We next claim that

$$I_x(x,t) \to 0 \quad \text{if } x \to \infty. \tag{4.19}$$

Indeed, otherwise we may assume that

$$I_x(x_n,t) \to M > 0$$

for a sequence $x_n \to \infty$. By (4.18) it follows that $I_x(x,t) > M/2$ if $|x - x_n| \leqslant \delta$, δ small, so that $I(x_n+\delta) - I(x_n-\delta) > M\delta/2$, a contradiction to (4.17).

From (4.17) and (4.19) we get, by integration,

$$(I_x + I)(x,t) = -\int_x^\infty (I_{xx} + I_x)\,dx$$

so that

$$|e^x I(x,t)|_x^\infty| \leqslant const. \int_x^\infty e^\xi \,d\xi \int_\xi^\infty e^{-(3+s)z}\,dz \cdot |\varphi|_{L^\infty}.$$

Recalling (4.17) we conclude that

$$|I(x,t)| \leqslant const.\, e^{(-3-s)x} |\varphi|_{L^\infty}.$$

Combining this with (4.13) and recalling (4.8), we conclude that estimate (4.6) holds also for $\sigma(x,t)$ where $x = x_1$ is as in (2.8).

5. The Hölder constant of $D_x^4\sigma(x,t)$ in x

Set

$$|\mu|_{b_T}^* = |\mu|_{L^\infty(b_T)} + \langle\mu\rangle_{x,b_T}^{(\alpha)} + \langle\mu\rangle_{t,b_T}^{(\beta)} + [\mu]_{b_T}^{(\alpha,\beta)}.$$

In this section we prove:

Lemma 5.1. *The following inequality holds*:

$$\left\langle e^{(s-\gamma)x} D_x^4 \sigma(x,t) \right\rangle_{x,b_T}^{(\alpha)} \leqslant const. \left(|\varphi|_{b_T}^* + \left|\varphi_x\right|_{b_T}^* \right), \tag{5.1}$$

where $\gamma = -3$.

In order to prove estimate (5.1) we use representation (4.8)–(4.10) for $\sigma(x,t)$ but rewrite $\sigma^0(x,t)$ more explicitly as in (3.37) with $\delta = 0$. Applying D_x to $\sigma^0(x,t)$ and recalling the definition of E, we get

$$D_x\sigma^0(x,t) = const. \int_0^t \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, e^{(\gamma-s)(x-\xi)} e^{-iyq(t-\tau,x-\xi)}$$
$$\times F\left(x-\xi, y, \tau\right) \int_{-\infty}^{\infty} dk_1\, E(k_1+ik_2, y)\, e^{i\gamma\xi(y-k)}, \tag{5.2}$$

where $F\left(x-\xi, y, \tau\right) = \varphi_\xi\left(x-\xi,\tau\right) - \varphi\left(x-\xi,\tau\right)(\gamma - s + i\gamma y)$, $k = k_1 + ik_2$; the factor $e^{i\gamma\xi y}$ comes from splitting $e^{-iyq(t-\tau,x)} = e^{-iyq(t-\tau,x-\xi)}\ e^{i\gamma\xi y}$.

Since the right-hand side of (5.2) is a convolution,

$$D_x^4\sigma^0(x,t) = const. \int_0^t \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, e^{(\gamma-s)(x-\xi)} e^{-iyq(t-\tau,x-\xi)}$$
$$\times F\left(x-\xi, y, \tau\right) \int_{-\infty}^{\infty} dk_1\, E(k_1+ik_2, y)\, (i\gamma(y-k))^3\, e^{i\gamma\xi(y-k)}$$
$$= const. \int_0^t \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, e^{(\gamma-s)\xi} e^{-iyq(t-\tau,x)}$$
$$\times F\left(\xi, y, \tau\right) \int_{-\infty}^{\infty} dk_1\, E(k_1+ik_2, y)\, (y-k)^3\, e^{-i\gamma(x-\xi)k}. \tag{5.3}$$

Here k_2 is a constant that may be taken to depend on ξ, and it will be chosen later on.

Remark 5.1. The inner integral in (5.3) is actually divergent. However if we divide the integrand by $(\Psi(y,k_1))^\delta$ where $\Psi(y,k_1) > 0$,

$$\Psi(y,k_1) \sim \begin{cases} |k_1| & \text{if} \quad |k_1| \to \infty, \quad |y| \quad \text{bounded}, \\ |y| & \text{if} \quad |y| \to \infty, \quad |k_1| \quad \text{bounded}, \\ |k_1| + |y| & \text{if} \quad |k_1| \to \infty, \quad |y| \to \infty \end{cases}$$

and δ is any small positive number, then the multiple integral on the right-hand side of (5.3) is convergent; this actually follows from the estimates of Section 4. Let us denote by $\left(D_x^4\sigma^0\right)_\delta$ the expression in (5.3) when the inner integral is multiplied by $(\Psi(y,k_1))^{-\delta}$. Similarly, we define $\left(D_x^j\sigma^0\right)_\delta$ by replacing $(y-k)^3$ by $(y-k)^{j-1}(i\gamma)^{j-4}(1+|k_1|)^\delta$. Then, if $\delta \to 0$ and $1 \leqslant j \leqslant 3$,

$$\left(D_x^j\sigma^0(x,t)\right)_\delta \to D_x^j\sigma^0(x,t)$$

pointwise. Hence, in the distribution sense,

$$D_x^4 \sigma^0 (x, t) = \lim_{\delta \to 0} \left(D_x^4 \sigma^0 (x, t) \right)_\delta .$$

It follows that (5.3) holds in the distribution sense provided we interpret the inner integral as

$$\lim_{\delta \to 0} \int_{-\infty}^{\infty} dk_1 E (k_1 + ik_2, y) \frac{(y-k)^3}{(\Psi (y, k_1))^\delta} e^{-i\gamma(x-\xi)k}.$$

We shall first estimate a function of the form

$$\begin{aligned}
\widehat{g} (x, t) &= e^{(s-\gamma)x} \int_0^t \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} dy \, H (y) \, e^{-iyq(t-\tau, x)} \\
&\quad \times \int_{-\infty}^{\infty} d\xi \, \mu \left(\xi, \tau\right) e^{(\gamma - s)\xi} \int_{-\infty}^{\infty} dk_1 \, E (k_1 + ik_2, y) \, (y-k)^3 \, e^{-i\gamma(x-\xi)k} \\
&= \int_0^t \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} dy \, H (y) \, e^{-iyq(t-\tau, x)} \int_{-\infty}^{\infty} d\xi \, \mu \left(\xi, \tau\right) e^{-(\gamma - s)(x-\xi)} \\
&\quad \times \int_{-\infty}^{\infty} dk_1 \frac{L (i (k_1 - y) - k_2)}{L (ik_1 - k_2 + 1)} (y - k_1 + ik_2)^3 \, e^{-i\gamma(x-\xi)(k_1 + ik_2)},
\end{aligned} \tag{5.4}$$

where $\mu (x, t)$ is a given function with finite norm $|\mu|_{b_T}^*$ and $\mu (x, 0) = 0$.

We introduce the function

$$d \left(\xi, y\right) = e^{i\gamma y} \int_0^{\infty} e^{i\gamma y \ln k_1} e^{-i\gamma \xi k_1} \, dk_1 \tag{5.5}$$

which can be written in the form

$$d \left(\xi, y\right) = d_1 \left(\xi, y\right) + d_2 \left(\xi, y\right),$$

where

$$\begin{aligned}
d_1 \left(\xi, y\right) &= e^{i\gamma y} \frac{e^{-i\gamma y \ln\left|\gamma \xi\right|}}{\left|\gamma \xi\right|} \int_0^{\infty} e^{i\gamma y \ln u} \cos u \, du, \\
d_2 \left(\xi, y\right) &= i e^{i\gamma y} \frac{e^{-i\gamma y \ln\left|\gamma \xi\right|}}{|\gamma| \, \xi} \int_0^{\infty} e^{i\gamma y \ln u} \sin u \, du,
\end{aligned} \tag{5.6}$$

as follows by the substitution $\left|\gamma \xi\right| k_1 = u$; $d_1 \left(\xi, y\right)$ is an even function in ξ and $d_2 \left(\xi, y\right)$ is an odd function in ξ.

As in Remark 5.1 we interpret the integral in (5.5) as a limit

$$\lim_{\delta\to 0}\int_0^\infty \frac{e^{i\gamma y\ln k_1}e^{-i\gamma\xi k_1}}{(\Psi(y,k_1))^\delta}\,dk_1,$$

and similarly interpret the integrals in (5.6). The same interpretation will be used in the following lemma.

Lemma 5.2. *Let*

$$p(y)=\int_0^\infty e^{i\gamma y\ln u}e^{iu}\,du\equiv p_1(y)+p_2(y),$$

where

$$p_1(y)=\int_0^\infty e^{i\gamma y\ln u}\cos u\,du,\quad p_2(y)=i\int_0^\infty e^{i\gamma y\ln u}\sin u\,du.$$

Then $p_1(y)$ *and* $p_2(y)$ *are twice continuously differentiable, and* $p_1(0)=0$.

Proof. We can write

$$p_1(y)=\int_0^\infty e^{i\gamma y\ln u}\cos u\,du=\int_0^\pi e^{i\gamma y\ln u}\cos u\,du+\lim_{\delta\to 0}\int_\pi^\infty e^{i\gamma y\ln u}\frac{\cos u}{u^\delta}\,du.$$

After two integrations by parts in the second summand we obtain

$$\begin{aligned}p_1(y)\;=\;&\int_0^\pi e^{i\gamma y\ln u}\cos u\,du+\int_\pi^\infty e^{i\gamma y\ln u}\left(\frac{\gamma^2y^2}{u^2}+\frac{i\gamma y}{u^2}\right)\cos u\,du\\&+e^{i\gamma y\ln\pi}\frac{i\gamma y}{\pi}.\end{aligned}\tag{5.7}$$

Next

$$p_2(y)=i\int_0^{\pi/2} e^{i\gamma y\ln u}\sin u\,du+\lim_{\delta\to 0}i\int_{\pi/2}^\infty e^{i\gamma y\ln u}\frac{\sin u}{u^\delta}\,du$$

and two integrations by parts in the second term yield

$$\begin{aligned}p_2(y)\;=\;&i\int_0^{\pi/2} e^{i\gamma y\ln u}\sin u\,du+i\int_{\pi/2}^\pi e^{i\gamma y\ln u}\frac{i\gamma y}{u}\cos u\,du\\&+i\int_\pi^\infty e^{i\gamma y\ln u}\left[\frac{\gamma^2y^2}{u^2}+\frac{i\gamma y}{u^2}\right]\sin u\,du.\end{aligned}\tag{5.8}$$

The assertions of the lemma now follow from formulas (5.7) and (5.8).

Lemma 5.3. *There holds, in the distribution sense,*

$$\int_0^\infty e^{-iyz}\,dz = \pi\delta(y) - \frac{i}{y}, \tag{5.9}$$

where $\delta(y)$ is the Dirac delta function.

Proof. We check equality (5.9) directly. From the definition of the inverse Fourier transform

$$F^{-1}\left(\widetilde{f}\right)(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(y)\,e^{iyz}\,dy \tag{5.10}$$

we get, in the distribution sense,

$$\begin{aligned}
F^{-1}\left(\pi\delta(y) - \frac{i}{y}\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(y) - \frac{i}{y}\right)e^{iyz}\,dy \\
&= \frac{1}{2} - \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\cos yz + i\sin yz}{y}\,dy \\
&= \frac{1}{2} + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin yz}{y}\,dy = \frac{1}{2} + \frac{1}{\pi}\cdot\frac{\pi}{2}\text{sign } z = \begin{cases} 1 & \text{for } z > 0, \\ 0 & \text{for } z < 0 \end{cases}
\end{aligned}$$

and from here (5.9) follows.

We now return to (5.4). As in Section 4 we will use a partition of the plane (y, k_1) and consider the part of the integral in (5.4) over the set $\widetilde{\Omega}_1 = \{(y, k_1) : y \in (-2M, 2M),\ k_1 \in (3M, \infty)\}$; we denote it by $g(x, t)$. In this set we have the asymptotic representation

$$\begin{aligned}
&\frac{L(i(k_1 - y) - k_2)}{L(ik_1 - k_2 + 1)}(y - k_1 + ik_2)^3 \\
&\quad = const.\, e^{-6\pi y} N(k_1, y)\, R(k_1, y)\,(k_1 - y)^3 \\
&\quad = const.\, e^{-i\gamma(k_1-y)\ln(k_1-y)+i\gamma k_1 \ln k_1}\left(e^{4(1+iy)} + O\left(\frac{1}{k_1}\right)\right),
\end{aligned} \tag{5.11}$$

where $k_1 O\left(\frac{1}{k_1}\right)$ is a smooth function of (y, k_1); here we used, in particular, the relation

$$R(k_1, y)(k_1 - y)^3 \approx 1 \quad \text{as} \quad k_1 \to \infty.$$

We have

$$\begin{aligned}(k_1-y)\ln(k_1-y)-k_1\ln k_1 &= -y\ln k_1+(k_1-y)\ln\left(1-\frac{y}{k_1}\right)\\ &= -y\ln k_1-y+O\left(\frac{1}{|k_1|}\right),\end{aligned}$$

as $|k_1|\to\infty$ so that, with $(y,k_1)\in\widetilde{\Omega}_1$, the main term in the k_1- integral on the right-hand side of (5.4) is

$$\begin{aligned}&\int_{3M}^{\infty}e^{i\gamma(y\ln k_1+y)}e^{-i\gamma\eta k_1}\,dk_1\\ &\quad=\int_0^{\infty}e^{i\gamma(y\ln k_1+y)}e^{-i\gamma\eta k_1}\,dk_1\\ &\qquad-\int_0^{3M}e^{i\gamma(y\ln k_1+y)}e^{-i\gamma\eta k_1}\,dk_1\quad\left(\eta=x-\xi\right);\end{aligned}\tag{5.12}$$

note that the second term here is a regular function of ξ. If we substitute this into the integral of $g\,(x,t)$ we obtain

$$g\,(x,t)=g_1\,(x,t)+\widetilde{g}\,(x,t)\,,$$

where

$$\begin{aligned}g_1\,(x,t) &= \int_0^t\frac{d\tau}{t-\tau}\int_{-2M}^{2M}dy\,H_1\,(y)\,e^{-iyq(t-\tau,x)}\\ &\quad\times\int_{-\infty}^{\infty}d\xi\,\mu\left(\xi,\tau\right)\,d\left(x-\xi,y\right)e^{-(\gamma-s)(x-\xi)}e^{\gamma k_2(x-\xi)},\end{aligned}$$

$H_1\,(y)=H\,(y)\,e^{4(1+iy)}$ (a bounded function), and $d\left(\xi,y\right)$ is the function defined in (5.5). In what follows, we prove that

$$\langle g_1\,(x,t)\rangle_{x,b_T}^{(\alpha)}\leqslant const.\;|\mu|_{b_T}^{*}\,.\tag{5.13}$$

The function $\widetilde{g}\,(x,t)$ arises from the regular term $O\,(1/\,|k_1|)$ in (5.11) and from the second term on the right-hand side of (5.12) (which is also regular). Hence the corresponding proof of (5.13) for $\widetilde{g}\,(x,t)$ is much simpler, and will be omitted.

Due to inequality (2.4) and restriction (3.30) on k_2, the value k_2 can be chosen such that, for some small $\varepsilon>0$,

$$-\left(\gamma-s\right)z+\gamma k_2 z=\left\{\begin{array}{ll}\varepsilon z, & z<0\\ -\varepsilon z, & z>0\end{array}\right\},\quad \varepsilon>0.$$

Then

$$g_1(x,t) = \int_0^t \frac{d\tau}{t-\tau} \int_{-2M}^{2M} dy\, H_1(y) \int_{-\infty}^{\infty} d\xi\, \mu\left(\xi,\tau\right)\, A\left(y,x,x-\xi,t-\tau\right), \tag{5.14}$$

where

$$A\left(y,x,\varsigma,\lambda\right) = e^{-iyq(\lambda,x)}\, d(\varsigma,y)\, e^{-\varepsilon|\varsigma|}. \tag{5.15}$$

We introduce the function

$$\left[\mu\right](x,y,\tau,t) = \mu(x,\tau) - \mu(y,\tau) - \mu(x,t) + \mu(y,t)$$

and rewrite (5.14) in the form

$$g_1(x,t) = \sum_{j=1}^{4} g_{1j}(x,t), \tag{5.16}$$

where

$$\begin{aligned} g_{11}(x,t) \;=\; & \int_0^t \frac{d\tau}{t-\tau} \int_{-2M}^{2M} dy\, H_1(y) \\ & \times \int_{-\infty}^{\infty} d\xi\, A\left(y,x,x-\xi,t-\tau\right) \left[\mu\right]\left(\xi,x,\tau,t\right), \end{aligned} \tag{5.17}$$

$$\begin{aligned} g_{12}(x,t) \;=\; & \int_0^t \frac{d\tau}{t-\tau} \int_{-2M}^{2M} dy\, H_1(y) \\ & \times \int_{-\infty}^{\infty} d\xi\, A\left(y,x,x-\xi,t-\tau\right) \left(\mu(x,\tau) - \mu(x,t)\right), \end{aligned} \tag{5.18}$$

$$\begin{aligned} g_{13}(x,t) \;=\; & \int_0^t \frac{d\tau}{t-\tau} \int_{-2M}^{2M} dy\, H_1(y) \\ & \times \int_{-\infty}^{\infty} d\xi\, A\left(y,x,x-\xi,t-\tau\right) \left(\mu\left(\xi,t\right) - \mu(x,t)\right), \end{aligned} \tag{5.19}$$

$$g_{14}(x,t) = \mu(x,t)\, N(x,t) \tag{5.20}$$

and

$$N(x,t)=\int_0^t \frac{d\tau}{t-\tau}\int_{-2M}^{2M} dy\, H_1(y)\int_{-\infty}^{\infty} d\xi\, A\left(y,x,x-\xi,t-\tau\right). \tag{5.21}$$

Lemma 5.4. *There holds*

$$|N(x,t)|\leqslant const. \tag{5.22}$$

Proof. We have by Lemma 5.3 and after the substitution $\ln\left(-\gamma^3(t-\tau)\right)=z$

$$\begin{aligned}\int_0^t \frac{d\tau}{t-\tau} e^{-iy\ln\left(-\gamma^3(t-\tau)\right)} &= \int_{-\infty}^{\ln\left(-\gamma^3 t\right)} e^{-iyz}\, dz\\ &= \int_{-\infty}^{0} e^{-iyz}\, dz + \int_0^{\ln\left(-\gamma^3 t\right)} e^{-iyz}\, dz\\ &= \pi\delta(y)+\frac{i}{y}+\frac{i}{y}\left(e^{-iy\ln\left(-\gamma^3 t\right)}-1\right),\end{aligned}$$

so that

$$\int_0^t \frac{d\tau}{t-\tau} e^{-iy\ln\left(-\gamma^3(t-\tau)\right)} = \pi\delta(y)+\frac{K(y,t)}{y}, \tag{5.23}$$

where $K(y,t)=ie^{-iy\ln\left(-\gamma^3 t\right)}$. Hence

$$N(x,t)=\int_{-2M}^{2M} dy\left(\pi\delta(y)+\frac{K(y,t)}{y}\right)H_1(y)\, e^{i\gamma yx}\int_{-\infty}^{\infty} d\xi\, e^{-\varepsilon|\xi|}\, d\left(\xi,y\right). \tag{5.24}$$

We decompose $d\left(\xi,y\right)$ into $d_1\left(\xi,y\right)+d_2\left(\xi,y\right)$ (as in (5.6)) and correspondingly decompose $N(x,t)$ into $N_1(x,t)+N_2(x,t)$. Since $d_2\left(\xi,y\right)$ is an odd function in ξ, $N_2(x,t)=0$. Since further $d_1\left(\xi,0\right)=0$ (by Lemma 5.2), we get

$$N(x,t)=\int_{-2M}^{2M} dy B(y)\, K(y,t)\, e^{i\gamma yx}\int_{-\infty}^{\infty} d\xi e^{-\varepsilon|\xi|}\frac{e^{-i\gamma y\ln|\xi|}}{|\xi|}, \tag{5.25}$$

where

$$B(y)=\frac{p_1(y)}{y}H_1(y)\, e^{i\gamma y}\frac{e^{-i\gamma y\ln|\gamma|}}{|\gamma|}. \tag{5.26}$$

Noting that

$$\int_0^1 d\xi \frac{e^{-i\gamma y \ln \xi}}{\xi} = \int_{-\infty}^0 e^{-i\gamma y z}\, dz = \pi\delta(y) + \frac{i}{\gamma y}$$

by Lemma 5.3, we can write the inner integral in (5.25) in the form

$$\begin{aligned} &2\int_0^\infty d\xi\, e^{-\varepsilon\xi} \frac{e^{-i\gamma y \ln \xi}}{\xi} \\ &= 2\left[\pi\delta(y) + \frac{i}{\gamma y} + \int_0^1 d\xi \left(e^{-\varepsilon\xi} - 1\right) \frac{e^{-i\gamma y \ln \xi}}{\xi} + \int_1^\infty d\xi\, e^{-\varepsilon\xi} \frac{e^{-i\gamma y \ln \xi}}{\xi}\right] \\ &= 2\left(\pi\delta(y) + \frac{i}{\gamma y} + b(y)\right), \end{aligned}$$

where $b(y)$ is a smooth function. It follows that

$$N(x,t) = 2\pi i p_1'(0)\, H_1(0) \frac{1}{|\gamma|} + 2\int_{-2M}^{2M} B(y)\, K(y,t)\, e^{i\gamma y x} \left(\frac{i}{\gamma y} + b(y)\right) dy. \quad (5.27)$$

The y-integrals in $\widehat{g}$, g_1 and g_{1j} can all be taken in the sense of principal value (p.v.) about $y = 0$ (by using the same arguments as in Remark 5.1), and the same then holds for the integral in (5.27). The main part of the integral in (5.27) is

$$p.v. \int_{-2M}^{2M} B(y)\, K(y,t)\, e^{i\gamma y x} \frac{i}{\gamma y}\, dy. \quad (5.28)$$

Recalling the definition of $K(y,t)$ (following (5.23)) and setting

$$v(x,t) = \ln\left(-\gamma^3 t\right) - \gamma x, \quad (5.29)$$

this integral takes the form

$$\frac{i}{\gamma} p.v. \int_{-2M}^{2M} B_1(y) \frac{e^{-iyv}}{y}\, dy, \quad (5.30)$$

where $B_1(y)$ is continuously differentiable. Writing $B_1(y) = (B_1(y) - B_1(0)) + B_1(0)$ and noting that $B_1(y) - B_1(0) = O(|y|)$ as $y \to 0$, we see that the main part of the last integral is equal to

$$\Phi(x,t) \equiv const. \int_{-2M}^{2M} B_1(0) \frac{\sin(y v(x,t))}{y}\, dy = const. \int_0^{2Mv(x,t)} \frac{\sin z}{z}\, dz. \quad (5.31)$$

We conclude that the integral in (5.30) is bounded and this completes the proof of Lemma 5.4.

Formulas (5.27)–(5.31) also enable us to prove Hölder continuity of $N(x,t)$. Indeed, using the definition of $v(x,t)$ and the estimate

$$|\Phi(x_1,t) - \Phi(x_2,t)| \leqslant const.\, |x_1 - x_2|,$$

we obtain

$$\langle N(x,t)\rangle^{(\alpha)}_{x,b_T} \leqslant const., \qquad \alpha \in (0,1). \tag{5.32}$$

6. The Hölder constant of $D_x^4\sigma(x,t)$ in x (continued)

We shall use Lemma 5.4 and (5.32) to estimate the Hölder continuity in x of the $g_{1j}(x,t)$ (defined in (5.17)–(5.20)).

We begin with $g_{11}(x,t)$. Using the inequality

$$\left|[\mu]\left(x,\xi,\tau,t\right)\right| \leqslant [\mu]^{(\alpha,\beta)}_{b_T} \left|x-\xi\right|^{\alpha} |t-\tau|^{\beta}, \tag{6.1}$$

we get

$$\begin{aligned} |g_{11}(x,t)| \leqslant{}& const.\, [\mu]^{(\alpha,\beta)}_{b_T} \int_0^t \frac{d\tau}{(t-\tau)^{1-\beta}} \int_{-2M}^{2M} dy\, |H_1(y)| \\ &\times \int_{-\infty}^{\infty} \frac{e^{-\varepsilon|x-\xi|}}{\left|x-\xi\right|^{1-\alpha}}\, d\xi \leqslant const.\, [\mu]^{(\alpha,\beta)}_{b_T}. \end{aligned} \tag{6.2}$$

We next estimate

$$\Delta_x g_{11}(x,t) = g_{11}(x_1,t) - g_{11}(x_2,t).$$

Setting $\Delta x = x_1 - x_2$ we can write

$$\begin{aligned} \Delta_x g_{11}(x,t) ={}& \int_0^t \frac{d\tau}{t-\tau} \int_{-2M}^{2M} dy\, H_1(y) \int_{|x_1-\xi|\leqslant 2|\Delta x|} d\xi\, A\left(y, x_1, x_1-\xi, t-\tau\right) \\ &\times [\mu]\left(\xi, x_1, \tau, t\right) - \int_0^t \frac{d\tau}{t-\tau} \int_{-2M}^{2M} dy\, H_1(y) \\ &\times \int_{|x_1-\xi|\leqslant 2|\Delta x|} d\xi\, [\mu]\left(\xi, x_2, \tau, t\right) A\left(y, x_2, x_2-\xi, t-\tau\right) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{d\tau}{t-\tau}\int_{-2M}^{2M} dy\, H_1(y) \\
&\times \int_{|x_1-\xi|>2|\Delta x|} d\xi\, [\mu](\xi, x_1, \tau, t)\left\{A\left(y, x_1, x_1-\xi, t-\tau\right)\right. \\
&\left. - A\left(y, x_2, x_2-\xi, t-\tau\right)\right\} + \int_0^t \frac{d\tau}{t-\tau}\int_{-2M}^{2M} dy\, H_1(y) \\
&\times \int_{|x_1-\xi|>2|\Delta x|} d\xi\, A\left(y, x_2, x_2-\xi, t-\tau\right)[\mu](x_2, x_1, \tau, t) \\
\equiv &\sum_{i=1}^{4} J_i.
\end{aligned}
\tag{6.3}
$$

Using (6.1), we get

$$
\begin{aligned}
|J_1| \leqslant\ & const.\, [\mu]_{b_T}^{(\alpha,\beta)} \int_0^t \frac{d\tau}{t-\tau}(t-\tau)^{\beta}\int_{-2M}^{2M} dy\, |H_1(y)| \\
&\times \int_{|x_1-\xi|\leqslant 2|\Delta x|} \left|x_1-\xi\right|^{\alpha} d\left(x_1-\xi, y\right) d\xi \\
\leqslant\ & const.\, [\mu]_{b_T}^{(\alpha,\beta)} \int_0^t \frac{d\tau}{t-\tau}(t-\tau)^{\beta}\int_{|x_1-\xi|\leqslant 2|\Delta x|} \left|x_1-\xi\right|^{\alpha-1} d\xi \\
\leqslant\ & const.\, [\mu]_{b_T}^{(\alpha,\beta)} \left|\Delta x\right|^{\alpha} t^{\beta}.
\end{aligned}
\tag{6.4}
$$

J_2 and J_4 can be estimated in a similar way, and J_3 is estimated by applying the mean value theorem. We conclude that

$$
|g_{11}(x_1, t) - g_{11}(x_2, t)| \leqslant const.\, [\mu]_{b_T}^{(\alpha,\beta)} \left|\Delta x\right|^{\alpha} T^{\beta}. \tag{6.5}
$$

We next turn to $g_{12}(x, t)$. We introduce the function $\psi(x, \tau) = \mu(x, \tau) - \mu(x, t)$ and note that

$$
\left|\psi(x, \tau)\right| \leqslant |\mu|_{b_T}^{*}\, |t-\tau|^{\beta} \tag{6.6}
$$

and

$$
\left|\psi(x_1, \tau) - \psi(x_2, \tau)\right| \leqslant |\mu|_{b_T}^{*}\, |x_1-x_2|^{\alpha}\, |t-\tau|^{\beta}. \tag{6.7}
$$

Proceeding as in Lemma 5.4 (but more simply because of the factor $|t-\tau|^{\beta}$) we find that

$$|g_{12}(x,t)| \leqslant const.\ |\mu|^{*}_{b_T}. \tag{6.8}$$

To prove Hölder continuity in x we note that, roughly speaking, if we apply Δ_x to $\mu(x,\tau) - \mu(x,t)$ we get (by virtue of (6.7)) the same estimate as in (6.5). On the other hand if we apply Δ_x to the other factor in the integrand in (5.18), i.e., to A, and use (6.6), we get the bound

$$const.\ |\mu|^{*}_{b_T}\ |x_1 - x_2|.$$

We need of course to first break the integral $\Delta_x g_{12}(x,t)$ into four integrals as in (6.3).

We next observe, by Lemma 5.4 and (5.32), that

$$|g_{14}(x,t)|_{L^{\infty}(b_T)} + \langle g_{14}(x,t)\rangle^{(\alpha)}_{x,b_T} \leqslant const.\ |\mu|^{*}_{b_T}. \tag{6.9}$$

Consider finally $g_{13}(x,t)$. To estimate it we use the decomposition

$$\mu\left(\xi,t\right) - \mu(x,t) = \left(\mu\left(\xi,t\right) - \mu(0,t)\right) + \left(\mu(0,t) - \mu(x,t)\right)$$

and correspondingly decompose g_{13} into $g_{131} + g_{132}$. Since

$$\left|\mu\left(\xi,t\right) - \mu(0,t)\right| \leqslant |\mu|^{*}_{b_T}\ \left|\xi\right|^{\alpha} \tag{6.10}$$

we can estimate $|g_{131}|$ as in the proof of Lemma 5.4 (in fact, the proof is now simpler due to bound (6.10)). We thus get

$$|g_{131}(x,t)| \leqslant const.\ |\mu|^{*}_{b_T} \tag{6.11}$$

and the same is true also for g_{132}, by Lemma 5.4.

Finally, to estimate $\Delta_x g_{13}(x,t)$ we proceed as in the case of $\Delta_x g_{12}(x,t)$ noting that Δ_x is applied to $\mu\left(\xi,t\right) - \mu(x,t)$ yields $\mu(x_1,t) - \mu(x_2,t)$.

We have thus proved that

$$|g_1(x,t)|_{L^{\infty}(b_T)} + \langle g_1(x,t)\rangle^{(\alpha)}_{x,b_T} \leqslant const.\ |\mu|^{*}_{b_T}, \tag{6.12}$$

and the same is true for $g(x,t)$, which is the part of the integral in (5.4) taken over the set $\widetilde{\Omega}_1$. Similarly one can estimate the integrals over the remaining sets Ω_j (see Fig. 1, Section 4). We conclude that

$$|\widehat{g}(x,t)|_{L^{\infty}(b_T)} + \langle \widehat{g}(x,t)\rangle^{(\alpha)}_{x,b_T} \leqslant const.\ |\mu|^{*}_{b_T}.$$

Comparing (5.3) with (5.4) we see that $D_x^4\sigma^0$ has the same form as the right-hand side of (5.4) with $\mu\left(\xi,\tau\right) = \ F\left(\xi,y,\tau\right)$. The fact that μ now depends on y causes just minor changes in the preceding analysis.

It remains to estimate $I\left(x,t\right)$ (see (4.8)–(4.10)). Denoting the right-hand side of (4.14) by $R\left(x,t\right)$, it suffices to estimate the Hölder coefficient of

$$\begin{aligned} e^{(s-\gamma)x}\frac{\partial^2 R}{\partial x^2} &= const.\, e^{(s-\gamma)x}\int_{-\infty}^{\infty} d\xi \frac{\partial}{\partial x}\left[\varphi\left(x-\xi,t\right)e^{(\gamma-s)(x-\xi)}\right]\frac{\partial N\left(\xi\right)}{\partial \xi} \\ &= const.\int_{-\infty}^{\infty} d\xi\left[\varphi_\xi\left(x-\xi,t\right)-(\gamma-s)\,\varphi\left(x-\xi,t\right)\right] \\ &\quad\times e^{-(\gamma-s)\xi}\frac{\partial N\left(\xi\right)}{\partial\xi}, \end{aligned} \tag{6.13}$$

where

$$N\left(\xi\right)=\int_{-\infty}^{\infty}\frac{e^{i\lambda\xi}\,d\lambda}{\left(i\lambda+2\right)\tan\left(i\lambda+2\right)\omega},\qquad \lambda=\lambda_1+i\beta_1$$

and $\beta_1 \neq 2 \pm \frac{\pi n}{\omega}$ so that the denominator does not vanish; β_1 is also restricted by (4.15). Note that

$$\tan\left(i\lambda_1+2-\beta_1\right)\omega=\begin{cases} +i+O\left(\frac{1}{|\lambda_1|}\right), & \lambda_1\to+\infty,\\[2ex] -i+O\left(\frac{1}{|\lambda_1|}\right), & \lambda_1\to-\infty.\end{cases}$$

It follows that

$$e^{\beta_1\xi}N\left(\xi\right)=i\int_0^{\infty}\frac{e^{i\lambda_1\xi}}{a+i\lambda_1}\,d\lambda_1-i\int_{-\infty}^{0}\frac{e^{i\lambda_1\xi}}{a+i\lambda_1}\,d\lambda_1+b\left(\xi\right),$$

where $a=2-\beta_1$ and $b\left(\xi\right)$, $b_\xi\left(\xi\right)$ are uniformly bounded functions. Suppose $\xi>0$. Substituting $\lambda_1\xi=\mu$ in the first integral and $\lambda_1\xi=-\mu$ in the second integral, we get

$$e^{\beta_1\xi}N\left(\xi\right)=i\int_0^{\infty}\left(\frac{e^{i\mu}}{a\xi+i\mu}-\frac{e^{-i\mu}}{a\xi-i\mu}\right)d\mu+b\left(\xi\right).$$

The real part of the integrand is zero and the imaginary part of the integral is

$$-2\int_0^{\infty}\frac{\mu\cos\mu}{a^2\xi^2+\mu^2}\,d\mu+2\int_0^{\infty}\frac{a\xi\sin\mu}{a^2\xi^2+\mu^2}\,d\mu\equiv K_1+K_2.$$

Clearly

$$K_1 = 2\ln\left|\xi\right| + b_1\left(\xi\right), \qquad K_2 = -2a\xi\ln\left|\xi\right| + b_2\left(\xi\right),$$

where $b_1\left(\xi\right)$, $b_2\left(\xi\right)$ and their first derivatives are bounded functions. One can derive this final formula also in case $\xi < 0$. It follows that

$$\frac{\partial}{\partial\xi}N\left(\xi\right) = e^{-\beta_1\xi}\left(\frac{2}{\xi} + b_3\left(\xi\right)\ln\left|\xi\right| + b_4\left(\xi\right)\right),$$

where $b_3\left(\xi\right)$, $b_4\left(\xi\right)$ are bounded functions. Substituting this into (6.13) and choosing β_1 appropriately for $\xi > 0$ and for $\xi < 0$, we can easily derive estimate (5.1) for I. This completes the proof of Lemma 5.1.

7. The Hölder constant of $D_x^4\sigma\left(x,t\right)$ in t

Lemma 7.1. *There holds*

$$\left\langle e^{(s-\gamma)x}D_x^4\sigma\left(x,t\right)\right\rangle_{t,b_T}^{(\beta)} \leqslant const.\left(\left|\varphi\right|_{b_T}^{*} + \left|\varphi_x\right|_{b_T}^{*}\right), \tag{7.1}$$

$$\left[e^{(s-\gamma)x}D_x^4\sigma\left(x,t\right)\right]_{b_T}^{(\alpha,\beta)} \leqslant const.\left(\left|\varphi\right|_{b_T}^{*} + \left|\varphi_x\right|_{b_T}^{*}\right). \tag{7.2}$$

To prove the lemma we use the notation of Section 5 but write $g_1\left(x,t\right)$ a little differently:

$$g_1\left(x,t\right) = \int_{-\infty}^{t}\frac{d\tau}{t-\tau}\int_{-2M}^{2M}dy\,H_1\left(y\right)\int_{-\infty}^{\infty}d\xi\,\mu\left(x-\xi,\tau\right)A\left(y,\xi,x,t-\tau\right),$$

here $\mu\left(x-\xi,\tau\right)$ was extended by 0 into $\tau < 0$. We further decompose $g_1\left(x,t\right)$ as in Section 5 by writing

$$\begin{aligned} g_1\left(x,t\right) &= \int_{-\infty}^{t}\frac{d\tau}{t-\tau}\int_{-2M}^{2M}dy\,H_1\left(y\right)\int_{-\infty}^{\infty}d\xi\,A\left(y,\xi,x,t-\tau\right) \\ &\quad\times\left\{\left[\mu\right]\left(x-\xi,x,\tau,t\right) + \left(\mu\left(x-\xi,t\right) - \mu\left(x,t\right)\right)\right. \\ &\quad\left.+\left(\mu\left(x,\tau\right) - \mu\left(x,t\right)\right) + \mu\left(x,t\right)\right\} \\ &= \sum_{j=1}^{4}g_{1j}\left(x,t\right). \end{aligned} \tag{7.3}$$

Let $\Delta t = t_1 - t_2 > 0$. Then

$$
\begin{aligned}
&[g_{13}(x,t_1) + g_{14}(x,t_1)] - [g_{13}(x,t_2) + g_{14}(x,t_2)] \\
&\quad = \int_{t_1-2\Delta t}^{t_1} M_1(x, t_1-\tau)\left[\mu(x,\tau) - \mu(x,t_1)\right] d\tau \\
&\qquad - \int_{t_1-2\Delta t}^{t_2} M_1(x, t_2-\tau)\left[\mu(x,\tau) - \mu(x,t_2)\right] d\tau \\
&\qquad + \int_{-\infty}^{t_1-2\Delta t} [M_1(x,t_1-\tau) - M_1(x,t_2-\tau)]\left[\mu(x,\tau) - \mu(x,t_2)\right] d\tau \\
&\qquad + L(x,t_1,t_2),
\end{aligned}
\tag{7.4}
$$

where

$$
\begin{aligned}
L(x,t_1,t_2) = & \int_{-\infty}^{t_1-2\Delta t} M_1(x,t_1-\tau)\left[\mu(x,t_2) - \mu(x,t_1)\right] d\tau \\
& + \int_{-\infty}^{t_1} M_1(x,t_1-\tau)\,\mu(x,t_1)\, d\tau - \int_{-\infty}^{t_2} M_1(x,t_2-\tau)\,\mu(x,t_2)\, d\tau
\end{aligned}
$$

and

$$
M_1(x,t-\tau) = \frac{1}{t-\tau}\int_{-2M}^{2M} dy\, H_1(y) \int_{-\infty}^{\infty} d\xi\, A\left(y,\xi,x,t-\tau\right). \tag{7.5}
$$

The function $L(x,t_1,t_2)$ we represent in the form

$$
\begin{aligned}
L(x,t_1,t_2) = & \ \mu(x,t_1)\int_{t_1-2\Delta t}^{t_1} M_1(x,t_1-\tau)\, d\tau + \mu(x,t_2)\int_{-\infty}^{t_1-2\Delta t} M_1(x,t_1-\tau)\, d\tau \\
& -\mu(x,t_2)\int_{-\infty}^{t_2} M_1(x,t_2-\tau)\, d\tau.
\end{aligned}
$$

Here in the first two integrals we change the variable $t_1 - \tau = z$ and in the third integral we introduce $t_2 - \tau = s$. Then

$$
\begin{aligned}
L(x,t_1,t_2) = & \ \mu(x,t_1)\int_0^{2\Delta t} M_1(x,z)\, dz + \mu(x,t_2)\int_{2\Delta t}^{\infty} M_1(x,z)\, dz \\
& -\mu(x,t_2)\int_0^{\infty} M_1(x,s)\, ds \\
= & \left[\mu(x,t_1) - \mu(x,t_2)\right]\int_0^{2\Delta t} M_1(x,z)\, dz.
\end{aligned}
$$

Each term in (7.4) can now be estimated by

$$const.\ \langle\mu\rangle_t^{(\beta)} \left|\Delta t\right|^{\beta}.$$

Indeed this can be done similarly to the derivation of the estimates in Sections 5 and 6.

Next we estimate

$$\begin{aligned}
&[g_{11}(x,t_1)+g_{12}(x,t_1)]-[g_{11}(x,t_2)+g_{12}(x,t_2)]\\
&\quad=\int_{t_1-2\Delta t}^{t_1} d\tau \int_{-2M}^{2M} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, [\mu]\left(x-\xi,x,\tau,t_1\right) \frac{A\left(y,\xi,x,t_1-\tau\right)}{t_1-\tau}\\
&\qquad-\int_{t_1-2\Delta t}^{t_2} d\tau \int_{-2M}^{2M} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, [\mu]\left(x-\xi,x,\tau,t_2\right) \frac{A\left(y,\xi,x,t_2-\tau\right)}{t_2-\tau}\\
&\qquad+\int_{-\infty}^{t_1-2\Delta t} d\tau \int_{-2M}^{2M} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, [\mu]\left(x-\xi,x,\tau,t_2\right)\\
&\qquad\times\left[\frac{A\left(y,\xi,x,t_1-\tau\right)}{t_1-\tau}-\frac{A\left(y,\xi,x,t_2-\tau\right)}{t_2-\tau}\right]\\
&\qquad+\int_{-\infty}^{\infty} d\xi\, [\mu]\left(x-\xi,x,t_1,t_2\right) \int_{-2M}^{2M} dy\, H(y) \int_0^{2\Delta t} \frac{A\left(y,\xi,x,z\right)}{z}\, dz.
\end{aligned}\tag{7.6}$$

Again by using arguments from Sections 5 and 6 we get that each term in (7.6) is estimated by

$$const.\ [\mu]^{(\alpha,\beta)} \left|\Delta t\right|^{\beta}.$$

Combining the above estimates we conclude that

$$|g_1(x,t_1)-g_1(x,t_2)| \leqslant const.\ |\mu|_{b_T}^{*} \left|\Delta t\right|^{\beta}$$

and this leads to estimate (7.1) for σ^0 as in the proof of Lemma 5.1. The corresponding proof for $I(x,t)$ follows as in Section 6.

It remains to prove (7.2), and we shall do this only for σ^0. We shall illustrate the proof only for the function of the form $g_{12}(x,t)$ from (5.18). So, let

$$\begin{aligned}
g_{12}(x,t) = &\int_{-\infty}^{t} d\tau \int_{-2M}^{2M} dy\, H_1(y) \frac{e^{-iy\ln\left(-\gamma^3(t-\tau)\right)}}{t-\tau}\\
&\times\int_{-\infty}^{\infty} d\xi\, d\left(\xi,y\right) e^{-\varepsilon|\xi|} e^{i\gamma y x} \left(\mu(x,\tau)-\mu(x,t)\right).
\end{aligned}$$

We can write the value $I = [g_{12}(x_1,t_1) - g_{12}(x_2,t_1)] - [g_{12}(x_1,t_2) - g_{12}(x_2,t_2)]$ in the form

$$
\begin{aligned}
I = & \int_{-\infty}^{t_1} d\tau \int_{-2M}^{2M} dy\, H_1(y)\, \frac{e^{-iy\ln\left(-\gamma^3(t_1-\tau)\right)}}{t_1-\tau} \int_{-\infty}^{\infty} d\xi\, d\left(\xi, y\right) e^{-\varepsilon|\xi|} \\
& \times \left[e^{i\gamma y x_1}\left(\mu(x_1,\tau) - \mu(x_1,t_1)\right) - e^{i\gamma y x_2}\left(\mu(x_2,\tau) - \mu(x_2,t_1)\right)\right] \\
& - \int_{-\infty}^{t_2} d\tau \int_{-2M}^{2M} dy\, H_1(y)\, \frac{e^{-iy\ln\left(-\gamma^3(t_2-\tau)\right)}}{t_2-\tau} \int_{-\infty}^{\infty} d\xi\, d\left(\xi, y\right) e^{-\varepsilon|\xi|} \\
& \times \left[e^{i\gamma y x_1}\left(\mu(x_1,\tau) - \mu(x_1,t_2)\right) - e^{i\gamma y x_2}\left(\mu(x_2,\tau) - \mu(x_2,t_2)\right)\right].
\end{aligned}
$$

Next, we represent this difference as follows:

$$
\begin{aligned}
I = & \int_{t_2-2\Delta t}^{t_1} d\tau \int_{-2M}^{2M} dy\, H_1(y)\, \frac{e^{-iy\ln\left(-\gamma^3(t_1-\tau)\right)}}{t_1-\tau} F(x_1,x_2,\tau,t_1) \\
& \times \int_{-\infty}^{\infty} d\xi\, d\left(\xi, y\right) e^{-\varepsilon|\xi|} - \int_{t_2-2\Delta t}^{t_2} d\tau \int_{-2M}^{2M} dy\, H_1(y) \\
& \times \frac{e^{-iy\ln\left(-\gamma^3(t_2-\tau)\right)}}{t_2-\tau} F(x_1,x_2,\tau,t_2) \int_{-\infty}^{\infty} d\xi\, d\left(\xi, y\right) e^{-\varepsilon|\xi|} \\
& + \int_{-\infty}^{t_2-2\Delta t} d\tau \int_{-2M}^{2M} dy\, H_1(y) \left[\frac{e^{-iy\ln\left(-\gamma^3(t_1-\tau)\right)}}{t_1-\tau} - \frac{e^{-iy\ln\left(-\gamma^3(t_2-\tau)\right)}}{t_2-\tau} \right] \\
& \times F(x_1,x_2,\tau,t_1) \int_{-\infty}^{\infty} d\xi\, d\left(\xi, y\right) e^{-\varepsilon|\xi|} \\
& + \int_{-\infty}^{t_2-2\Delta t} d\tau \int_{-2M}^{2M} dy\, H_1(y)\, \frac{e^{-iy\ln\left(-\gamma^3(t_2-\tau)\right)}}{t_2-\tau} \\
& \times F(x_1,x_2,t_2,t_1) \int_{-\infty}^{\infty} d\xi\, d\left(\xi, y\right) e^{-\varepsilon|\xi|},
\end{aligned}
\tag{7.7}
$$

where

$$
F(x_1,x_2,\tau,t) = e^{i\gamma y x_1}\left[\mu\right](x_1,x_2,\tau,t) + \left(e^{i\gamma y x_1} - e^{i\gamma y x_2}\right)\left(\mu(x_2,\tau) - \mu(x_2,t)\right),
$$

and note that

$$
|F(x_1,x_2,\tau,t)| \leqslant const. \left(\left[\mu\right]^{(\alpha,\beta)} + \langle\mu\rangle_t^{(\beta)}\right) |x_1-x_2|^{\alpha} |t-\tau|^{\beta}. \tag{7.8}
$$

The last integral in (7.7) is estimated immediately by writing

$$\int_{-\infty}^{t_2-2\Delta t} d\tau \int_{-2M}^{2M} dy\, H_1(y) \frac{e^{-iy\ln\left(-\gamma^3(t_2-\tau)\right)}}{t_2-\tau} \int_{-\infty}^{\infty} d\xi\, d\left(\xi,y\right) e^{-\varepsilon|\xi|}$$
$$= \int_0^{\infty} dz \int_{-2M}^{2M} dy\, H_1(y) \frac{e^{-iy\ln\left(-\gamma^3 z\right)}}{z} \int_{-\infty}^{\infty} d\xi\, d\left(\xi,y\right) e^{-\varepsilon|\xi|}$$
$$- \int_0^{2\Delta t} d\tau \int_{-2M}^{2M} dy\, H_1(y) \frac{e^{-iy\ln\left(-\gamma^3 z\right)}}{z} \int_{-\infty}^{\infty} d\xi\, d\left(\xi,y\right) e^{-\varepsilon|\xi|}$$

and applying Lemma 5.4 (or, rather, a slightly different version of it).

Recalling, by the formulas derived in the proof of Lemmas 5.2–5.4, that

$$\int_{-\infty}^{\infty} d\xi\, d\left(\xi,y\right) e^{-\varepsilon|\xi|}$$
$$= \left(\int_{-\infty}^{\infty} d\xi\, d_2\left(\xi,y\right) e^{-\varepsilon|\xi|} = 0\right)$$
$$= 2e^{i\gamma y} p_1(y) \left\{ \int_0^1 d\xi \frac{e^{i\gamma y\ln\xi}}{\xi} e^{-\varepsilon|\xi|} + \int_1^{\infty} d\xi \frac{e^{i\gamma y\ln\xi}}{\xi} e^{-\varepsilon|\xi|} \right\}$$
$$= \left\{ \pi\delta(y) - \frac{i}{\gamma y} + \int_0^1 d\xi \frac{e^{i\gamma y\ln\xi}}{\xi} \left(e^{-\varepsilon|\xi|} - 1\right) + \int_1^{\infty} d\xi \frac{e^{i\gamma y\ln\xi}}{\xi} e^{-\varepsilon|\xi|} \right\}$$
$$\times 2e^{i\gamma y} p_1(y),$$

and using (7.8), we can also estimate without difficulties the first three integrals on the right-hand side of (7.7), and thus obtain the bound

$$|[g_2(x_1,t_1) - g_2(x_2,t_1)] - [g_2(x_1,t_2) - g_2(x_2,t_2)]|$$
$$\leqslant const. \left([\mu]^{(\alpha,\beta)} + \langle\mu\rangle_t^{(\beta)}\right) |x_1-x_2|^{\alpha} |t_1-t_2|^{\beta}.$$

This leads as before to the completion of the proof of (7.2).

8. Construction of the solution

In Sections 4–7 we have established estimates for the function $\sigma(x,t)$ defined by (3.26). Taking its Fourier transform $\tilde{\sigma}(\lambda,t)$, we arrive at (3.24), and if we then take the Laplace transform, $\sigma^*(\lambda,v)$, we arrive at (3.23), which leads to (3.20), and finally to (3.4).

From (3.4) we deduce that the inverse Laplace transform of $v\sigma^*(\lambda - 3i, v)$ exists, and it is the function $\partial\tilde{\sigma}/\partial t$ given by (3.3). Hence

$$\frac{1}{\lambda\mu}\left[\frac{\partial\tilde{\sigma}(\lambda - i, t)}{\partial t} - \tilde{f}(\lambda - i, t)\right] = -(\lambda + 2i)(\lambda + i)\tanh\lambda\omega\tilde{\sigma}(\lambda + 2i, t). \tag{8.1}$$

Let us first assume that

$$f_{x_1}, f_{x_1 x_1} \in C^{1+\alpha,\beta,\alpha}(b_T). \tag{8.2}$$

By integration by parts in the Fourier transform of f we then get

$$|\tilde{f}(\lambda - i, t)| \leqslant \frac{C}{|\lambda - i|^3}. \tag{8.3}$$

Then, from (3.24) and the properties of $H(y)$ in (3.25) it follows that

$$|\tilde{\sigma}(\lambda + 2i, t)| \leqslant \frac{C}{|\lambda + 2i|^{6-3\delta}} \tag{8.4}$$

for any small $\delta > 0$. Using this in (8.1), we get

$$\left|\frac{\partial\tilde{\sigma}}{\partial t}(\lambda - i, t) - \tilde{f}(\lambda - i, t)\right| \leqslant \frac{C}{|\lambda - i|^{3-3\delta}}. \tag{8.5}$$

Consider the function

$$u(x_1, x_2, t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\partial\tilde{\sigma}}{\partial t}(\lambda - i, t) - \tilde{f}(\lambda - i, t)\right]\frac{\cosh\lambda x_2}{\lambda\mu\sinh\lambda\omega}e^{i\lambda x_1}\,d\lambda. \tag{8.6}$$

From (8.5) it follows that the integral converges and, by differentiation,

$$\frac{\partial^2 u}{\partial x_1} + \frac{\partial^2 u}{\partial x_2} = 0, \quad -\omega < x_2 < 0,$$

$$\left.\frac{\partial u}{\partial x_2}\right|_{x_2=0} = 0.$$

Next, from (8.5) and (8.3) we have

$$\left|\frac{\partial\tilde{\sigma}}{\partial t}(\lambda - i, t)\right| \leqslant \frac{C}{|\lambda - i|^{3-3\delta}}.$$

Hence we can write, at $x_2 = -\omega$,

$$\begin{aligned}\frac{\partial\sigma}{\partial t} - \mu e^{x_1}\frac{\partial u}{\partial x_2} &= \frac{\partial\sigma}{\partial t} - e^{x_1}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\partial\tilde{\sigma}}{\partial t}(\lambda - i, t) - \tilde{f}(\lambda - i, t)\right]e^{i\lambda x_1}\,d\lambda \\ &= \frac{\partial\sigma}{\partial t} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial\tilde{\sigma}}{\partial t}(\lambda - i, t)e^{i(\lambda - i)x_1}\,d\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\lambda - i, t)e^{i(\lambda - i)x_1}\,d\lambda \\ &= f(x_1, t).\end{aligned}$$

We finally verify the boundary condition (2.11). From (8.6) and (8.1) we have

$$u(x_1, -\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\lambda + 2i)(\lambda + i)\tilde{\sigma}(\lambda + 2i, t)e^{i\lambda x_1}\,d\lambda. \tag{8.7}$$

Substituting $\lambda + 2i = q$ and using estimate (8.4), we find that the right-hand side of (8.7) is equal to

$$-e^{2x_1}\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1} + 1\right)\sigma(x_1, t).$$

We have thus proved that (u, σ) forms a solution of (2.9)–(2.12) provided f satisfies the additional condition (8.2). In the general case, instead of (8.3) we only have the inequality

$$|\tilde{f}(\lambda - i, t)| \leqslant \frac{C}{|\lambda - i|},$$

so that

$$|\tilde{\sigma}(\lambda + 2i, t)| \leqslant \frac{C}{|\lambda + 2i|^{4-3\delta}}.$$

Nevertheless (recalling (8.1)) the integral in (8.6) is still convergent. Furthermore, by approximating f by functions satisfying (8.2) we deduce that u satisfies (2.9) and (2.10)–(2.12) in the weak sense. By elliptic regularity, these relations then also hold in the classical sense (we use here the already established regularity of σ). This completes the proof of Theorem 2.1 in the special case (2.7).

In order to prove the theorem in the general case, we introduce the solution W to

$$\begin{aligned}\Delta_y W &= f_0 \quad \text{in } G_T, \\ W &= f_1 \quad \text{on } g_T, \\ \frac{\partial W}{\partial y_2} &= 0 \quad \text{on } y_2 = 0,\end{aligned}$$

and set $v = u - W$. Then v is a solution of (2.1) as in the special case (2.7) with f replaced by

$$\overline{\overline{f}} = f - \mu \frac{\partial W}{\partial n}. \tag{8.8}$$

We claim that

$$\| \overline{\overline{f}} \|_{E_s^{1+\alpha,\beta,\alpha}} \text{ is bounded by the right-hand side of (2.6).} \tag{8.9}$$

Indeed, let $w = e^{(s+1)x_1} W(x_1, x_2, t)$ (here, as before, we abuse notation, by adopting the change of variable (2.8) while maintaining the same notation for W). Then

$$\begin{aligned} \mathcal{L}_{s+1} w &\equiv \Delta_x w - 2(s+1) \frac{\partial w}{\partial x_1} + (s+1)^2 w = e^{(s-1)x_1} f_0 \quad \text{in } B_T, \\ w &= e^{(s+1)x_1} f_1 \quad \text{on } b_T, \\ \left. \frac{\partial w}{\partial x_2} \right|_{x_2=0} &= 0. \end{aligned}$$

We can then apply to $w - e^{(s+1)x_1} f_1$ Theorems 6.4.1.1 and 6.4.1.3 of Grisvard [6] to deduce that

$$\|w\|_{C_x^{2+\alpha}} \leqslant const. \{\|e^{(s+1)x_1} f_1\|_{C_{x_1}^{2+\alpha}} + \|e^{(s-1)x_1} f_0\|_{C_x^{\alpha}}\}.$$

A similar inequality can be obtained for $(w(x, t_1) - w(x, t_2))/|t_1 - t_2|^{\beta}$, and together they establish assertion (8.9).

The function $\overline{\overline{f}}$, however, does not vanish near $x_1 = -\infty$ (or near $r = \infty$). But if we introduce

$$\overline{w} = e^{-\ell x_1} W \quad (\text{for any } \ell > 0)$$

so that

$$\begin{aligned} \mathcal{L}_{-\ell} \overline{w} &= e^{-(\ell+2)x_1} f_0 \quad \text{in } B_T, \\ \overline{w} &= e^{-\ell x_1} f_1 \quad \text{on } b_T, \\ \left. \frac{\partial \overline{w}}{\partial x_2} \right|_{x_2=0} &= 0, \end{aligned}$$

and apply to $\overline{w}$ Theorems 6.4.1.1, 6.4.1.3 of Grisvard [6] (recalling that f_0, f_1 vanish if $x_1 < x_1^0$ for some $x_1^0 < 0$), we find that $W(x_1, x_2, t)$ decreases exponentially to any order, as $x_1 \to -\infty$, together with its $D_x^{2+\alpha}$ derivatives and its β-Hölder coefficient in t; more precisely,

$$e^{-\ell x_1} W \in C^{2+\alpha,\beta,\alpha}(B_T \cap \{x_1 < 0\})$$

for any $\ell > 0$.

This allows us to carry out the proof of Theorem 2.1 in the special case (2.7), as before; indeed, a much milder decrease of f, as $x_1 \to -\infty$, is needed in the proof.

Remark 8.1. The assumption made in Theorem 2.1 that f_0, f and f_1 vanish for $|y|$ large enough may be replaced by the assumption that these functions decrease exponentially to an appropriate order with their respective derivatives and local Hölder coefficients in t.

Remark 8.2. Let $f(x_1,t) \in E_{s,-m}^{1+\alpha,\alpha,\beta}(g_T) = E_s^{1+\alpha,\alpha,\beta}(g_T) \cap E_{-m}^{1+\alpha,\alpha,\beta}(g_T)$, $s>0$, $m>0$. Note that a function $f(x_1,t)$ from $E_s^{1+\alpha,\alpha,\beta}(g_T)$ which satisfies condition (2.14) belongs to $E_{s,-m}^{1+\alpha,\alpha,\beta}(g_T)$ with any m. Repeating the proof of Lemma 4.1 we find that $|\sigma(x_1,t)| \leqslant const.\, e^{(-3+m)x_1}$ if we take $3k_2 < 3-m$ for $\xi < 0$ and $3k_2 > 3-m$ for $\xi > 0$ in representation (4.2) with $\delta = 0$ in order to ensure the convergence of the integral with respect to ξ. Recall that, by (3.29) with $\delta = 0$, k_2 must also satisfy

$$\max\left(2 - \frac{\pi}{\omega}, -1 - \frac{\pi}{2\omega}\right) < 3k_2 < \min\left(5 + \frac{\pi}{2\omega}, 2 + \frac{\pi}{\omega}\right).$$

For $\omega < \pi/2$ there exists an $\eta > 0$ such that for $m = 3 + \eta$ the conclusion of Lemma 4.1 is true and $|\sigma(x_1,t)| \leqslant const.\, e^{\eta x_1}$. Hence, as $r \to \infty$,

$$\sigma(r,t) = O\left(\frac{1}{r^\eta}\right) \quad \text{in } g_T. \tag{8.10}$$

It then also follows that, as $r \to \infty$,

$$\begin{aligned} u(r,\theta,t) &= O\left(\frac{1}{r^\eta}\right) \quad \text{in } G_T, \\ \nabla u(r,\theta,t) &= O\left(\frac{1}{r^{1+\eta}}\right) \quad \text{in } G_T, \\ \sigma_r(r,t) &= O\left(\frac{1}{r^{1+\eta}}\right), \qquad \sigma_t(r,t) = O\left(\frac{1}{r^{1+\eta}}\right) \quad \text{in } g_T. \end{aligned} \tag{8.11}$$

Corresponding estimates can be obtained for general functions f_0, f, f_1 as in (2.2) and (2.3).

Theorem 8.1. *Under the assumptions of Theorem* 2.1, *if* $\omega < \frac{\pi}{2}$ *then there exists a unique solution of* (2.1) *in the class of functions* (u, σ) *satisfying* (2.5) *and estimates* (8.10) *and* (8.11) *for* $r \to \infty$.

Proof. We have to prove that if $f_0 \equiv 0$, $f \equiv 0$, $f_1 \equiv 0$ in (2.1) then $u \equiv 0$, $\sigma \equiv 0$. By integration by parts

$$\begin{aligned}\int_{G_T} |\nabla u|^2 &= -\int_{G_T} u\Delta u + \int_{g_T} u\frac{\partial u}{\partial n} = \mu^{-1}\int_{g_T} \sigma_{rr}\sigma_t \\ &= \mu^{-1}\int_{g_T}\left[\frac{\partial}{\partial r}(\sigma_r\sigma_t) - \sigma_r\frac{\partial \sigma_r}{\partial t}\right] dr\, dt \\ &= \mu^{-1}\int_0^T \sigma_r\sigma_t\Big|_{r=0}^{r=\infty} dt - \frac{\mu^{-1}}{2}\int \sigma_r^2\Big|_{t=0}^{t=T} dr\end{aligned}$$

or

$$\int_{G_T} |\nabla u|^2 + \frac{\mu^{-1}}{2}\int_g \sigma_r^2(\cdot, T) = \mu^{-1}\int_0^T \sigma_r\sigma_t\Big|_{r=0}^{r=\infty}; \tag{8.12}$$

here we used the boundary conditions of (2.1) and the fact that

$$\int_{r=R} u\frac{\partial u}{\partial n} \to 0 \quad \text{if } R \to \infty \qquad \text{by (8.11)).}$$

From (8.11) it also follows that $\sigma_r\sigma_t \to 0$ as $r \to \infty$. Since also $\sigma_r\sigma_t = 0$ at $r = 0$ (by the regularity assumptions of (2.5)), we deduce that the right-hand side of (8.12) vanishes. This implies that $u \equiv 0$, $\sigma \equiv 0$.

9. Proof of Theorem 2.1

All we need to prove is that

$$\left\| D_x^3\sigma \right\|_{C_{s+2}^{\alpha, \frac{1+\alpha}{3}}(g_T)} + \left\| D_x^2\sigma \right\|_{C_{s+1}^{\alpha, \frac{2+\alpha}{3}}(g_T)}$$

is finite and is bounded by the right-hand side of (2.6).

It will be useful to motivate the proof by considering first the model problem for the classical Hele-Shaw problem (1.1)–(1.4) in a smooth initial domain as in Bazaliy

[1]. We denote by $\sigma^*(\lambda, v)$ the Laplace transform (in t) of the Fourier transform (in x) of $\sigma(x,t)$, and similarly define $f^*(\lambda, v)$. Then by (8)–(10) of [1] for the model problem,

$$\sigma^*(\lambda, v) = \frac{f^*(\lambda, v)}{v + |\lambda|^3}$$

and, similarly,

$$(\sigma_x)^*(\lambda, v) = \frac{(f_x)^*(\lambda, v)}{v + |\lambda|^3}; \tag{9.1}$$

the corresponding relation for the Hele-Shaw model problem in an angular domain is (3.4).

We now observe that for the Cauchy problem of the heat equation

$$\begin{aligned} u_t - u_{xx} &= \varphi(x,t) \quad \text{in} \quad -\infty < x < \infty, \quad t > 0, \\ u \ |_{t=0} &= 0 \end{aligned}$$

there is the representation

$$u^*(\lambda, v) = \frac{\varphi^*(\lambda, v)}{v + |\lambda|^2},$$

and this relation can be used to derive the well-known estimate

$$\|u\|_{C_{x,t}^{2+\alpha,(2+\alpha)/2}} \leqslant C \|\varphi\|_{C_{x,t}^{\alpha,\alpha/2}}. \tag{9.2}$$

By analogy, if in (6.1) $f_x \in C_{x,t}^{\alpha,\alpha/3}$ then it should be true that

$$\sigma_x \in C_{x,t}^{3+\alpha,(3+\alpha)/3}, \tag{9.3}$$

and thus

$$\sigma_{xx} \in C_{x,t}^{2+\alpha,(2+\alpha)/3}, \ \sigma_{xxx} \in C_{x,t}^{1+\alpha,(1+\alpha)/3},$$

where the space $C^{k+\alpha,(k+\alpha)/3}$ is similar to the space $C^{k+\alpha,(k+\alpha)/2}$ that is used in the second-order parabolic theory. Indeed, this was proved in [1].

We now need to establish a similar result that consists in the inclusions

$$D_x^3\sigma \in C_{s+2}^{\alpha, \frac{1+\alpha}{3}}(g_T) \cap E_s^{1+\alpha,\alpha/3,\alpha}(g_T),$$

$$D_x^2\sigma \in C_{s+1}^{\alpha, \frac{2+\alpha}{3}}(g_T) \cap E_{s+1}^{2+\alpha,\alpha/3,\alpha}(g_T).$$

We represent $\sigma(x,t)$ in the convolution form (3.26) and calculate $D_x^3\sigma(x,t)$ by applying one derivative to the integrand $f\left(x-\xi,\tau\right)e^{\gamma(1-\delta)(x-\xi)}$ and two derivatives to the inner integral with respect to λ. After some transformations we arrive at an integral of the form (cf. (5.3))

$$\begin{aligned} D_x^3\sigma(x,t) = \mathit{const.} \int_{-\infty}^{t} \frac{d\tau}{(t-\tau)^{1-\delta}} \int_{-\infty}^{\infty} dy\, H(y) \int_{-\infty}^{\infty} d\xi\, e^{(-3(1-\delta)-s)\xi} e^{-iyq(t-\tau,x)} \\ \times F\left(\xi,y,\tau\right) \int_{-\infty}^{\infty} dk_1\, E\left(k_1+ik_2,y,\delta\right)(y-k)^2 e^{-i\gamma(x-\xi)k}. \end{aligned} \tag{9.4}$$

As in Sections 4,5 we use a partition of the (y,k_1) -plane into a finite number region and consider the part of the integral in (9.4) taken over the domain

$$\widetilde{\Omega}_1 = \{(y,k_1) : y \in (-2M,2M),\ k_1 \in (3M,\infty)\}.$$

In this domain

$$E\left(k_1+ik_2,y,\delta\right)(y-k)^2 \approx \mathit{const.}\, e^{i\gamma(y\ln k_1+y)} \frac{k_1^2}{k_1^{3(1-\delta)}} e^{4(1+iy)} \quad \text{as } k_1 \to \infty \tag{9.5}$$

since, from (3.34), $R\left(k,y,\delta\right) \approx \mathit{const.}/k_1^{3(1-\delta)}$ as $k_1 \to \infty$.

If we take $\delta = 1/3$ (or $\delta = 2/3$) then we need to modify the contour of integration in (3.20) in order to avoid the double pole of $G(z+\varsigma)$ at $z+\varsigma = 1/3$ (or $z+\varsigma = 2/3$). Accordingly, in (y,k_1)-integrals, if $\delta = 1/3$ (or 2/3) we must exclude the values $k_2 = 0, \pm 1/3$.

If we take $\delta = 1/3$ then the asymptotic behavior of $E\left(k_1+ik_2,y,\delta\right)(y-k)^2$ as $k_1 \to \infty$ is given by the right-hand side of (5.11) (where we have estimated $D_x^4\sigma^0(x,t)$). Therefore in order to estimate the Hölder constant of $D_x^3\sigma(x,t)$ with respect to t we can use the same arguments as in Section 7, but in the integral in τ we replace the function $(t-\tau)^{-1}$ by the function $(t-\tau)^{-1+1/3}$. This leads to the inequality

$$\left\langle e^{(s+2)x} D_x^3\sigma(x,t)\right\rangle_{t,b_T}^{((1+\alpha)/3)} \leqslant \mathit{const.}\left(|\varphi|_{b_T}^{*} + \left|\varphi_x\right|_{b_T}^{*}\right); \tag{9.6}$$

the exponent $e^{(s+2)x}$ comes from $e^{(-3(1-\delta)-s)\xi}$ with $\delta = 1/3$.

In a similar way, by using the integral representation with $\delta = 2/3$ we obtain the estimate

$$\left\langle e^{(s+1)x} D_x^2 \sigma(x,t) \right\rangle_{t,b_T}^{((2+\alpha)/3)} \leqslant const. \left(|\varphi|_{b_T}^* + \left|\varphi_x\right|_{b_T}^* \right). \tag{9.7}$$

To get the Hölder constant of $D_x^3 \sigma(x,t)$ and $D_x^2 \sigma(x,t)$ with respect to x we use the arguments of Sections 5 and 6. Thus altogether we obtain the estimate

$$\left\| D_x^3 \sigma \right\|_{C_{s+2}^{\alpha, \frac{1+\alpha}{3}}(g_T)} + \left\| D_x^2 \sigma \right\|_{C_{s+1}^{\alpha, \frac{2+\alpha}{3}}(g_T)} \leqslant const. \left(|\varphi|_{b_T}^* + \left|\varphi_x\right|_{b_T}^* \right). \tag{9.8}$$

We have thus shown that in the model problem we have an additional regularity of a solution with respect to t and this completes the proof of Theorem 2.1.

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